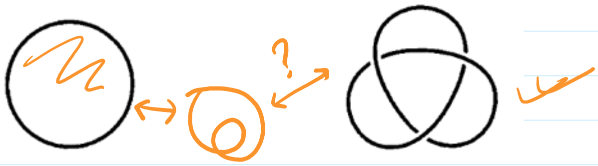
Not Theory (Sad I couldn't make the pun)

Wednesday, 4 September, 2024 03:10 PM

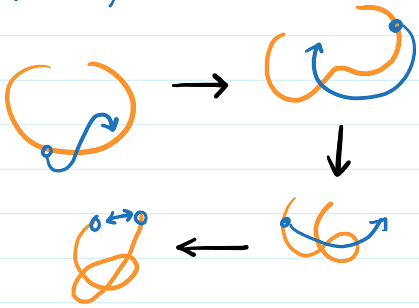
**KNOT** → loop of string.

(Take a string, tie a knot and glue the ends)

(Assumptions) String has no thickness, i.e., cross section is a single point.



Unknot $K_0$ Trefoil knot.



How do we know these are different?

Different, in sense that given one knot we cannot twist & turn the knot to transform it into the other one, i.e. without using scissors and glue. (called equivalent).

Very hard. (Haken, 1961)

Proved it is an NP problem

TotDHaken

(Unknot Theorem) It is decidable wheather a given knot, is equivalent to the unknot.

HISTORY (TKB-A

Aether

Knot in

Much of the early interest in knot theory was motivated by chemistry. In the 1880s, it was believed that a substance called ether pervaded all of space. In an attempt to explain the different types of matter, Lord Kelvin (William Thomson, 1824-1907) hypothesized that atoms were merely knots in the fabric of this ether. Different knots would then correspond to different elements (Figure 1.7).

$p_1 \cdots p_n$

A $k$ # — # $k_n$

Finite

New

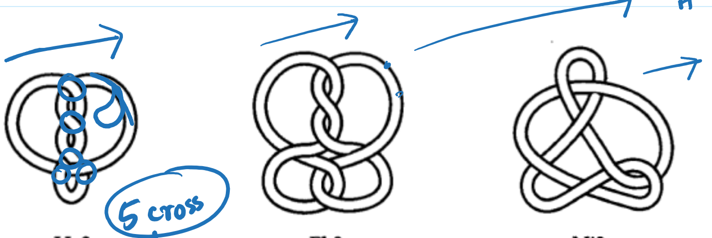



*Figure 1.7* Atoms are knotted vortices?

6

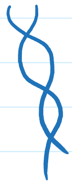
This convinced the Scottish physicist Peter Guthrie Tait (1831-1901) that if he could list all of the possible knots, he would be creating a table of the elements. He spent many years tabulating knots. At the same time, an American mathematician named C. N. Little was working on his own tabulations for knots.

Well, he was wrong.

Michelson-Morley experiment demonstrated there was no ether. ⇒ Losing life's work + Interest in knots.


BIOCHEMISTS discovered knotting in DNA

+ Properties of knotted molecules depend on the type of knots.



(For More details, read Chapter 7 of TKB-A).

Back to Mathematics,

KNOT THEORY is a subbranch of Topology.

We need it!

**Topology** (from the Greek words τόπος, 'place, location', and λόγος, 'study') is the branch of mathematics concerned with the properties of a geometric object that are preserved under continuous deformations, such as stretching, twisting, crumpling, and bending; that is, without closing holes, opening holes, tearing, gluing, or passing through itself.

From <https://en.wikipedia.org/wiki/Topology>

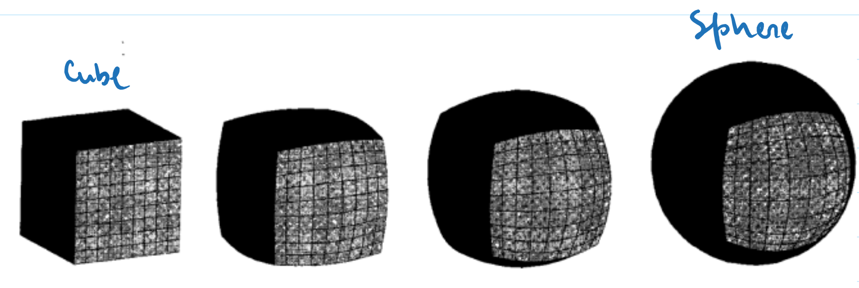
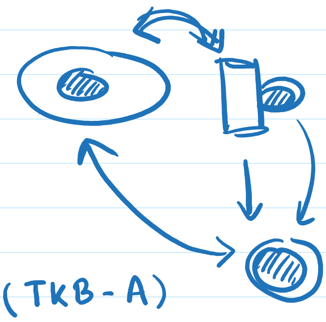



*Figure 1.8* A cube and a sphere are the same in topology.



(TKB-A)

By homeomorphism,

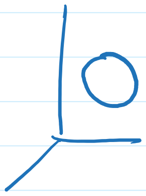
In mathematics and more specifically in topology, a **homeomorphism** (from Greek roots meaning "similar shape", named by Henri Poincaré),[2][3] also called **topological isomorphism**, or **bicontinuous function**, is a bijective and continuous function between topological spaces that has a continuous inverse function. Homeomorphisms are the isomorphisms in the category of topological spaces—that is, they are the mappings that preserve all the topological properties of a given space. Two spaces with a homeomorphism between them are called **homeomorphic**, and from a topological viewpoint they are the same.

From <https://en.wikipedia.org/wiki/Homeomorphism>

Mathematically,

A homeomorphism is a bijective fn. f from topological spaces $X \longrightarrow Y$ such that:

1) f is continuous.



spaces $X \longrightarrow Y$ such that :

i) $f$ is continuous.

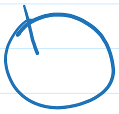
ii) $\exists$ a continuous inverse $g: f(X) \longrightarrow X$ s.t. $g \cdot f = f \cdot g = Id$.

$\therefore$ Representing knots as embedding $\longrightarrow$

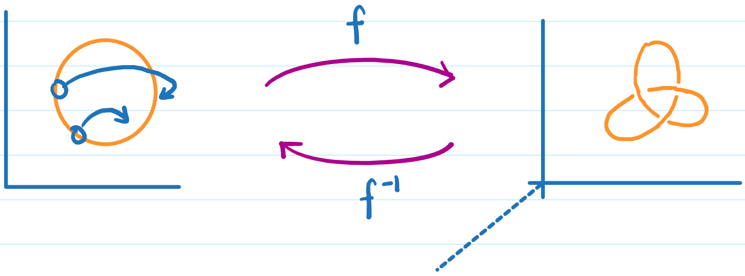
$\hookrightarrow$ homeomorphism of a set on its image.

A knot is an embedding of $S^1$ on $\mathbb{R}^3$.

$S^1$: circle.



$k_1 \leftrightarrow k_2$




Returning back to our

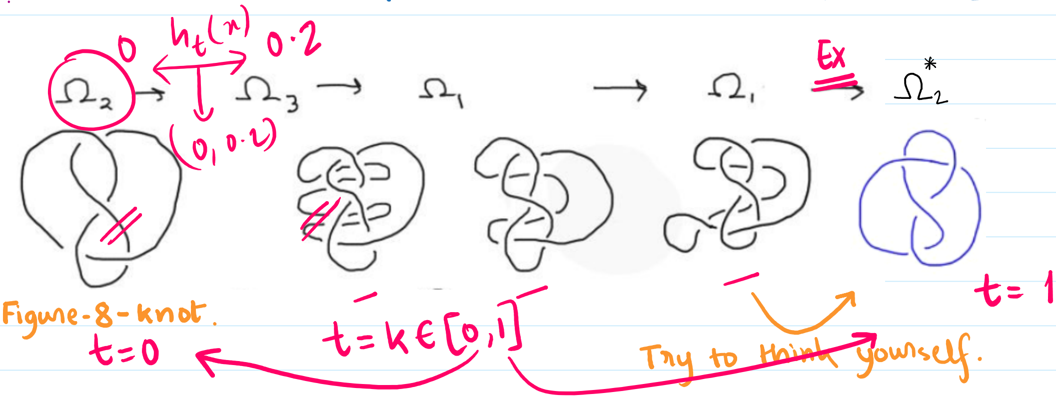
**Main Q**: How do we differentiate b/w 2 knots?

To know that, first we have to define when 2 knots are equivalent

**Defn**: Two knots $k_1, k_2$ are similar if $\exists$ a continuous mapping $h: \mathbb{R}^3 \times [0,1] \longrightarrow \mathbb{R}^3$, s.t. $\forall t \in [0,1]$, the maps $h_t : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by $h_t(x) = h(x,t)$ are homeomorphisms and $h_0 = Id$, $h_1(k_1) = k_2$.

Ambient $\Rightarrow$ Isotopy.

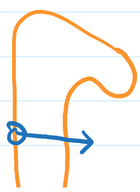
Planar Isotopy.



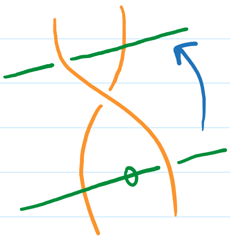

Back to Practical Sense $\longrightarrow$

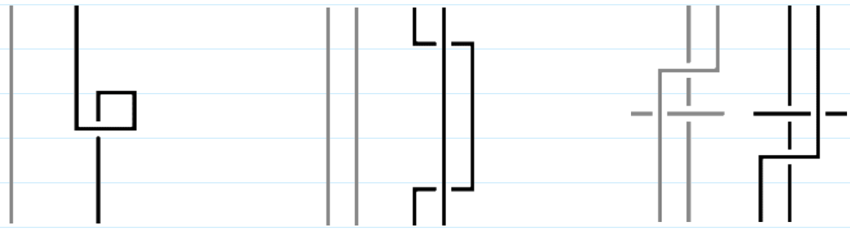
The theorem is mathematically precise. But it's hard to understand and somehow the feel behind this whole equivalency is lost behind the terminologies.

Same happened for Kurt Reidemeister.
He formulated moves (operations) on knots.



the formulated moves (operations) on knots.





Type-I (twist) Type-II (poke) Type-III (slide).

**Theorem)** (Reidemeister) Two knots are equivalent iff one can be transformed into another by finitely many Reidemeister moves.

## Composition of Knots. (#) TKA-

Given 2 knots $K_1, K_2$ ; we can construct a new knot $K_1 \# K_2$ by removing arcs from the 2 knots and joining them together.

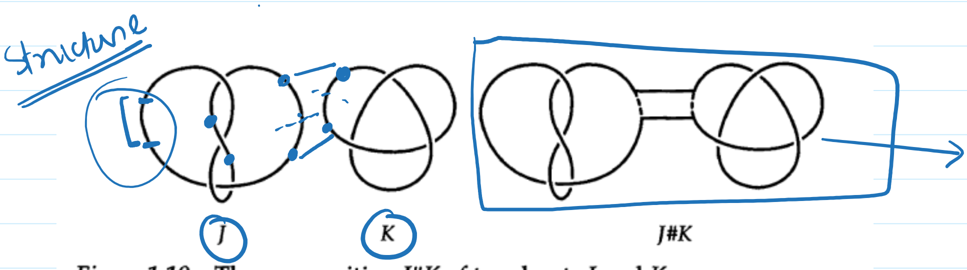



*Figure 1.10* The composition $J\#K$ of two knots $J$ and $K$.

The only condition here, is to choose the arcs which are outside, hence donot have crossings. (sneaky little ...)

$K \longrightarrow \cdot$ $K_1, K_2$ $\qquad K_1 \# K_2 = K.$

↳ composite knots

$\textcircled{K} = K \# \boxed{U}$ $\quad 1$

$= K \# A \# B$ → not factori

Unknot

$K \longrightarrow \nexists\, K_1, K_2$ s.t. $K_1 \# K_2 = K$

↳ prime knot

unknot

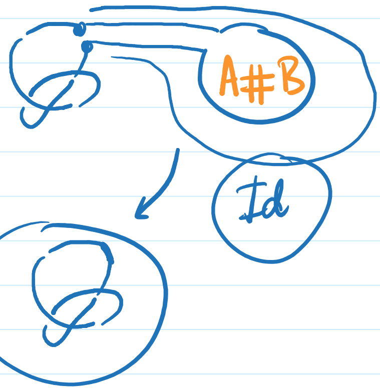
$7 = 7 \times 1$

Id.

$K_0 = \mathcal{K}_1 \# \mathcal{K}_2$

A#B

Id



$K = K \# K_0$

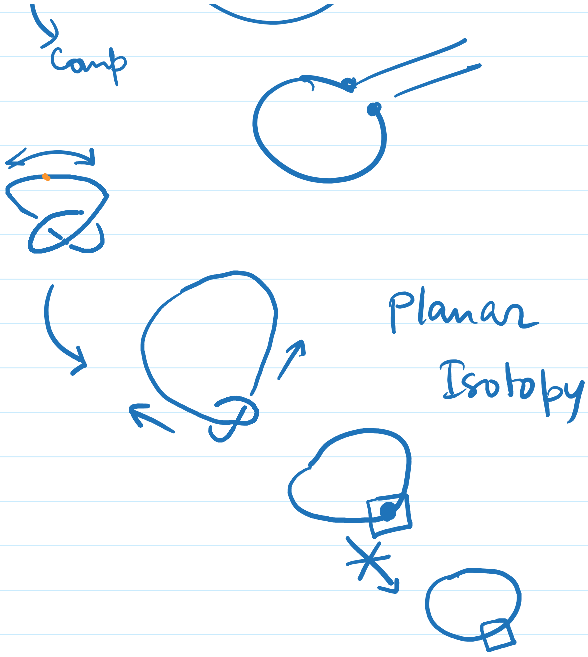
$= K \# \mathcal{K}_1 \# \mathcal{K}_2$

Comp

$= K \# \ldots$

$K = \boxed{K_1 \# K_2}$

Unique




Terminologies → Composite, Factor knots, Id = Trivial knot, Prime knots

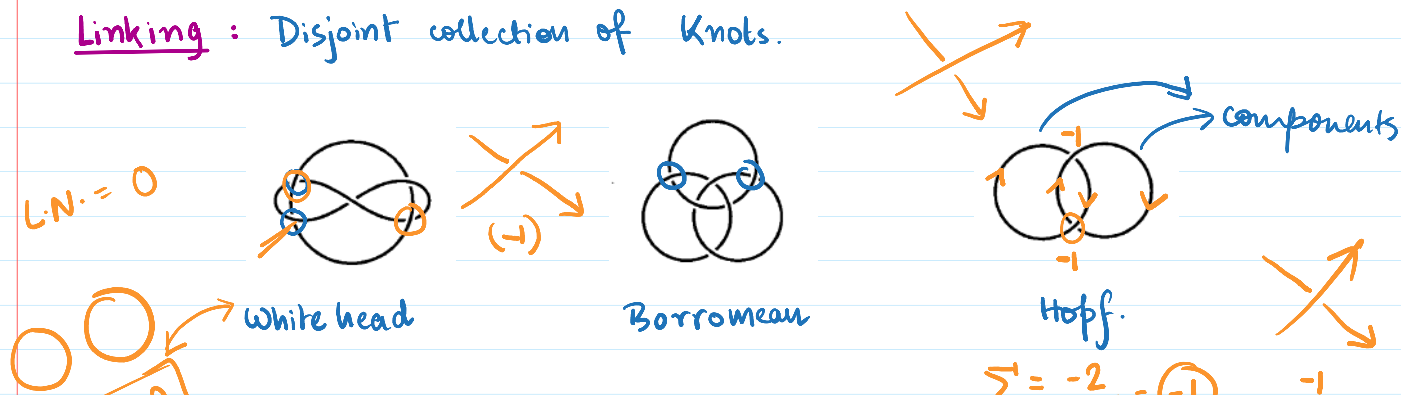
| |
|---|
| **Observation**: Unknot is not composite. Otherwise, every knot is composite ? |

Unique Prime Factorisation, Orientation.
(Read the composite section from TKB-A).

Are Reidemeister moves sufficient?
Like we still cannot say if 2 knots are equivalent, what if finite means following a specific order till $10^{723}$?

We will be needing a quantity which doesn't change under the Reidemeister moves, in other words, an invariant.

**Linking**: Disjoint collection of Knots.




$\Sigma = \frac{-2}{2} = -1$

$\boxed{L.N. = 0}$

Reidemeister moves hold purpose.

→ Splitable if components of links can be deformed to lie on different sides of a plane.

Consider the quantity of linking number.

different sides of a plane.

Consider the quantity of linking number, defined as →

- Take a link with components P, Q.
- Start by orienting them.
- Compute a +1 for case 1 and −1 for case 2 shown

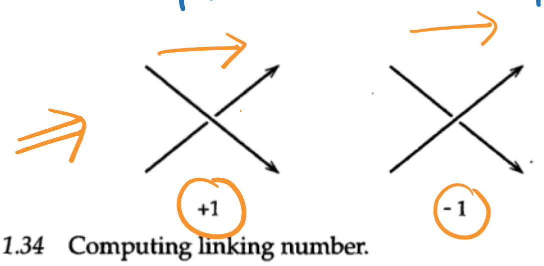



*Figure 1.34* Computing linking number.

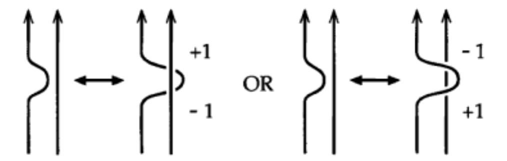
↻ Rotations allowed

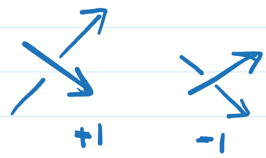
- Add everything and divide by 2.
  (Take absolute value if reqd.)

**Claim:** L·N doesn't change under Reidemeister moves.

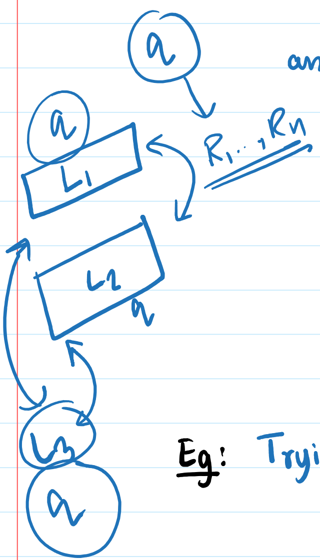
**Pf:** Under type I, we twist or untwist a self loop, thus only a single component is involved ⇒ No change in L·N.

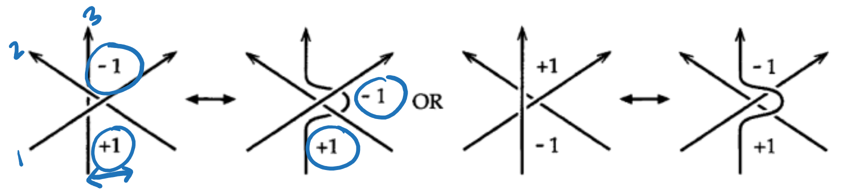
Under type II,




and type III,

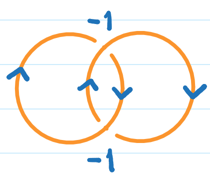


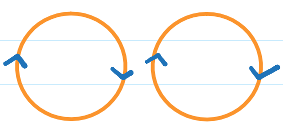




LN is an invariant under R. Moves.

the L·N. remains constant. ▨

**Eg:** Trying to compare the Hopf link and Unlink,



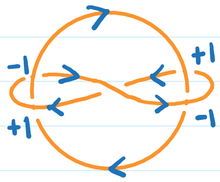
→ Splittable

$L\cdot N. = (-1-1)/2 = -1$  $L\cdot N. = 0$

Since LN are different, we say Hopf link and Unlink are not equivalent at all.

Can we consider L·N. as our invariant?

Sadly, NO!




$$L\cdot N\cdot = \frac{(-1-1+1+1)}{2} = 0$$

which is same as the unlink. ($\Rightarrow\Leftarrow$).

Hence we still need an invariant,
as we never found one!

$K_1$   $K_2$   $K_2$   $\boxed{RM}$

Link

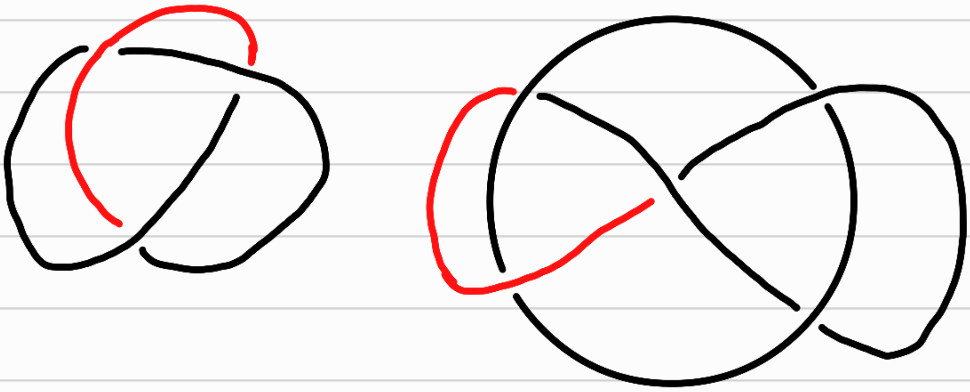
T   Chromatic

1) Tricolourability

# Knot Theory (Lecture-2)

## Colorings (Ralph Fox)

☆ Strand: A peice of link that goes from one undercrossing to another with only overcrossings in between.



Examples of a strand

# Tricolorable: A "Knot diagram" is called tricolorable if each strand can be drawn using three colors, say R(red), Y(Yellow), B(Blue) or for practical sense colors denoted by {0,1,2} (as we would eventually like to generalise to more than three colors).
While also following the below mentioned conditions.

1) Atleast two of the colors are used.
2) At any crossing either only one color is used or all colors are used. (never 2)

Acceptable

Not acceptable

☆ **Invariant:** Property of a knot which can't be lost (or gained is implied) with change in representation. [Knot diagrams]

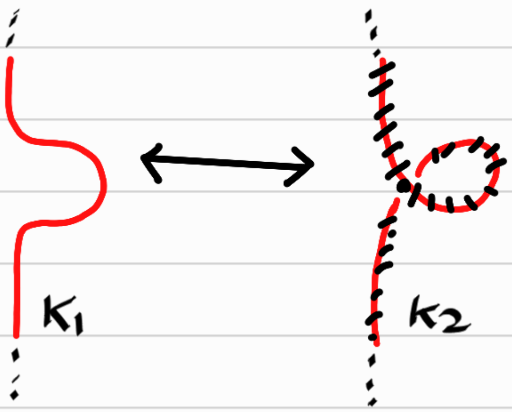
Now some of our great observers must have realized where we are heading with all of this?

Yes!! Tricolorability is an invariant

→ Recall if two knot diagrams represent the same knot then ∃ a series of Reidemeister moves taking one knot diagram to another.

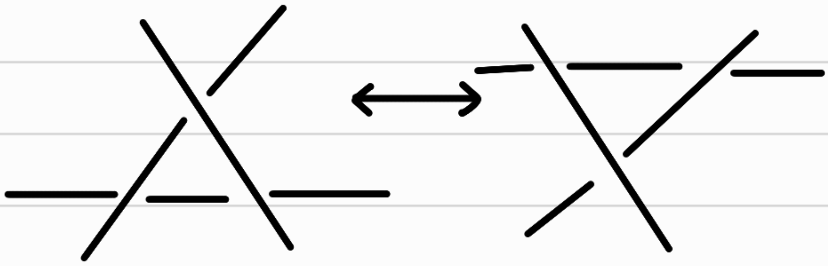
⇒ Enough to show Reidemeister moves preserve tricolorability (as again ∵ Reidemister moves always "work both ways", no creation is clearly implied)

□ R-I
(twist)

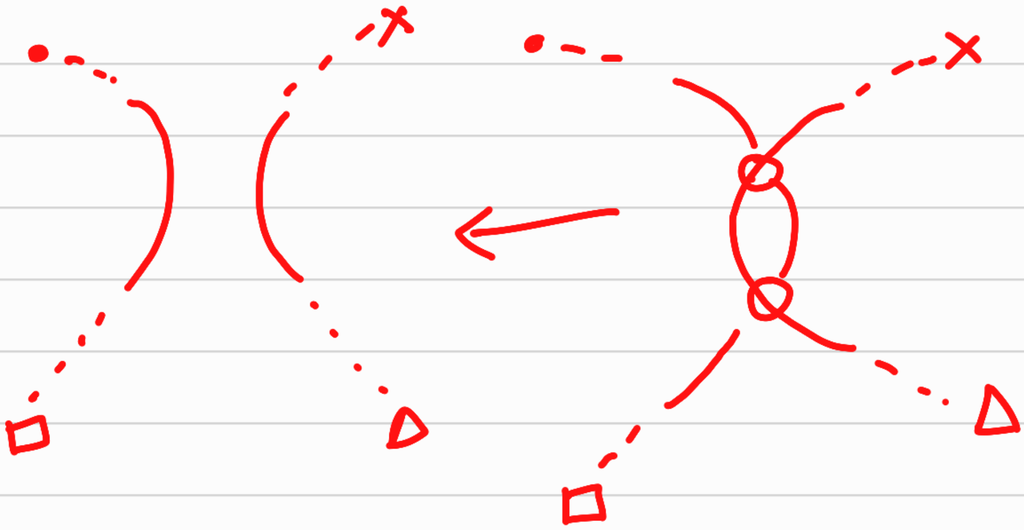


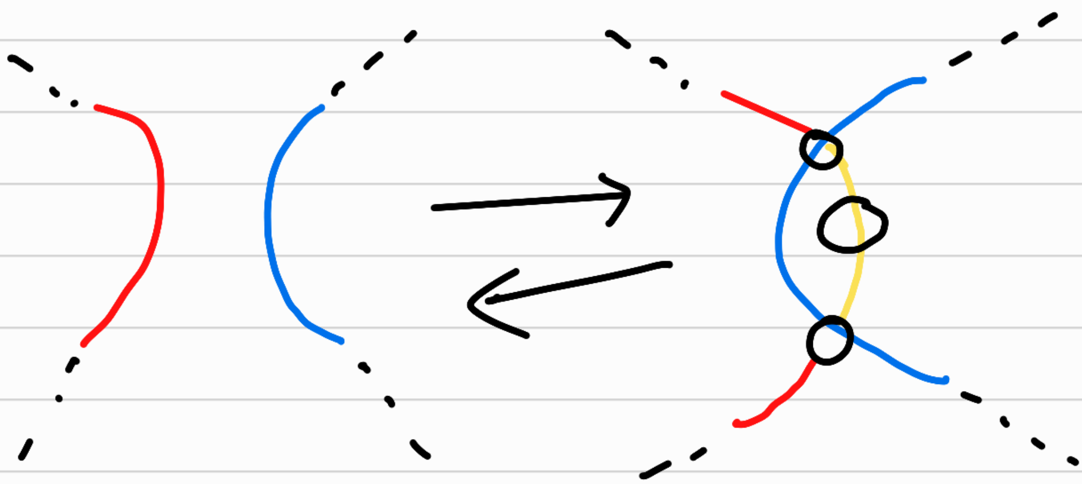

Say either one of $K_1$ or $K_2$ is colorable, then the parts shown above must be composed of one single color if in the other representation we color the corresponding parts with same colors and leave the rest of the colorings unchanged, then we can clearly observe that this is a valid coloring for the other diagram. Hence the other knot diagram will also stay colorable.

☐ R-III ("Slide") (H.W., not very trivial. Attendees are encouraged to share their attempts in the group)

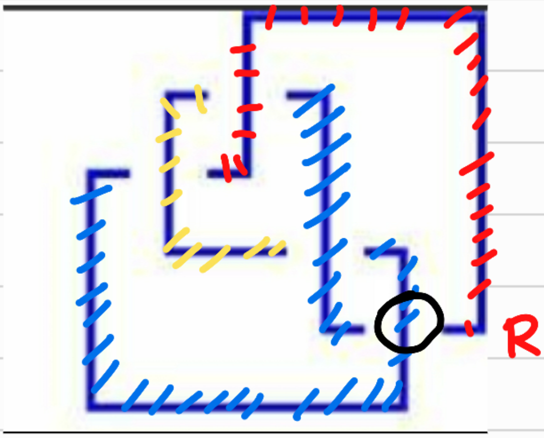


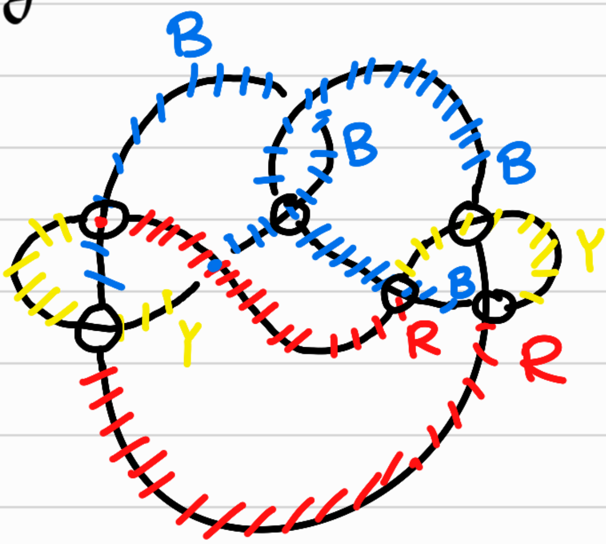
☐ R-II ("poke") (In class discussion)

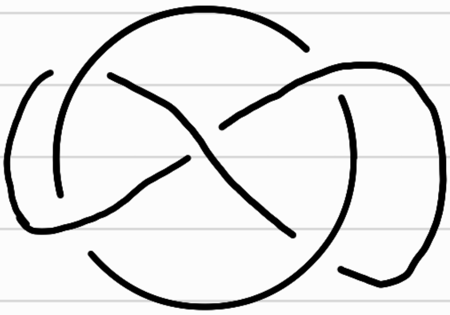


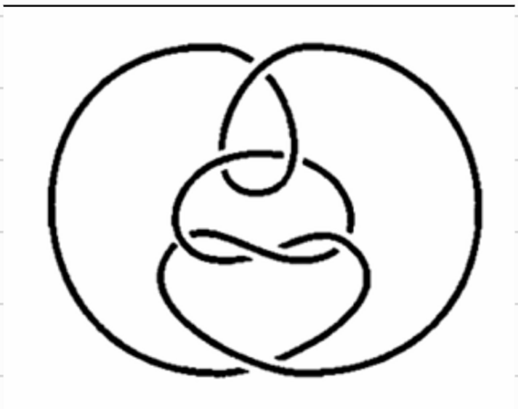


Q.) Check for tricolorability

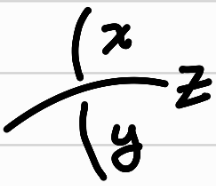




Mathematical way of restating condⁿ two :-




$x+y \equiv 2z \pmod 3$

Proof: Clearly,

$x+y+z \equiv 0 \pmod 3$

$\Rightarrow x+y+z \equiv 3z \pmod 3$

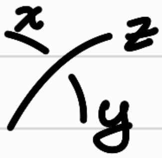
Now as fellow mathematicians, what is the one thing we all love to do? GENERALIZE

• What is so special about 3? NOTHING MUCH REALLY

P.T.O.

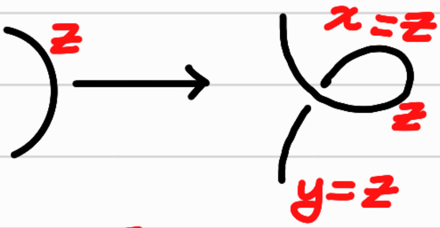
☆We say that a knot is p-colorable if:
(prime $p \geq 3$) (Why prime? (Why not 2?)

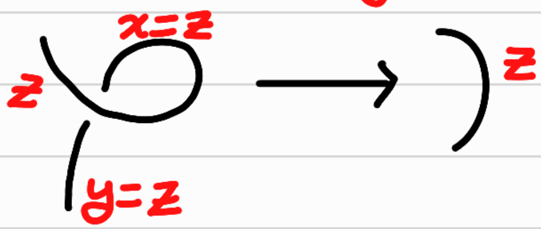
1) We use atleast two labels from $\{0, 1, 2, \ldots, p-1\}$

2) At each crossing  $x+y \equiv 2z \pmod{p}$

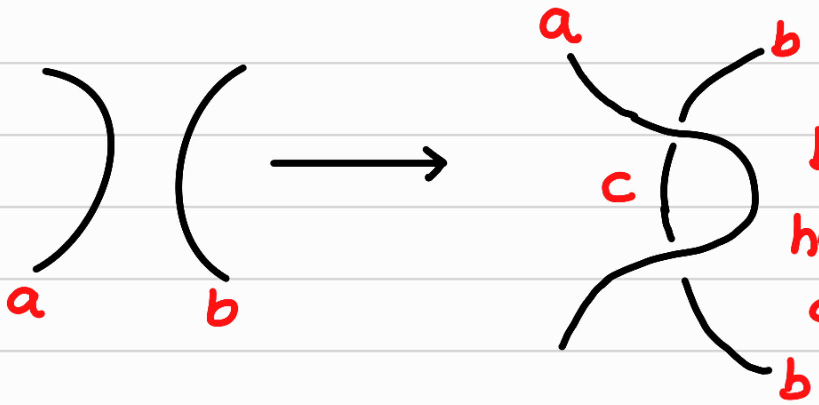
Again enough to show R. moves preserve p-colorability
$\forall$ prime $p \geq 3$. (Basic congruence from NT required)

☐ R-I

1)  $x+y \equiv 2z \ (p)$
$y \equiv z \ (p)$

2) 

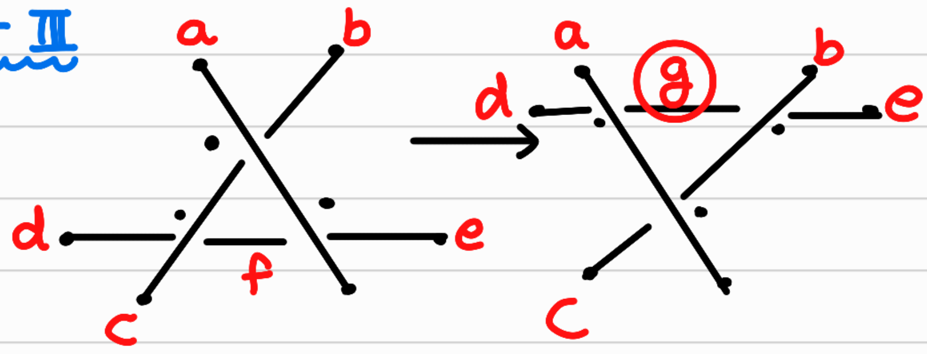
☐ R-II

1) 

$b+c \equiv 2a \ (p)$
has a sol$^n$ for c always

2) Works on default as shown before

☐ R-III



① $-b+c \equiv 2a\,(p)$ To show:
② $-d+f \equiv 2c\,(p)$ $\boxed{2a-d \equiv 2b-e}\ (p)$
③ $-f+e \equiv 2a\,(p)$

① + ② − ③

$$2a-d \equiv 2b-e\ (p)$$
$$c-d \equiv b-e\ (p)$$
$$c-d+e-b \equiv 0\ (p)$$

$b+c$
$+d+\cancel{f}$ $\equiv 2c\ (p)$
$-\cancel{f}-e$
$\Rightarrow b-c+d-e \equiv 0\ (p)$

Answering previous questions, there is no reason why we can't extend colorability to all composites $\geq 3$.

☆ A special case for 2 is that no knot is 2-colourable, as $x+y \equiv 2z\ (2) \Rightarrow x \equiv y\,(2) \Rightarrow x\ \&\ y$ have the same color for all under-crossing pairs. By "moving around" the knot we would observe that all strands will be forcefully attached with the same color as our first choice thus violating cond$^n$ 1.

Why we are mostly interested in primes? Wait for now!

☆ Figuring out if a knot is p-colorable manually is a lot of pain!

What are good ways to store information? maybe a matrix b/w crossings and arcs (strands) Bingo!!

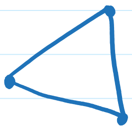
# Knots in a different way

Saturday, 30 November, 2024 10:44 AM

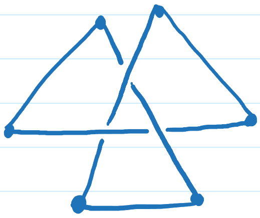
## Sticks

4 sticks / 5 sticks ✗.

6 sticks



Unknot
(u.s.)



trefoil knot
(6 sticks)

$s(k)$, $c(k)$

$$s(k) \leq 2c(k)$$

## Tabulating knots

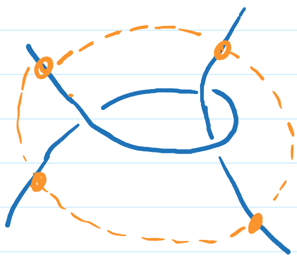
Dowker, Conway.

Planar Graphs.

## Conway Notation

'Tangles'

Building blocks
for knot.



NW
SW
NE
SE

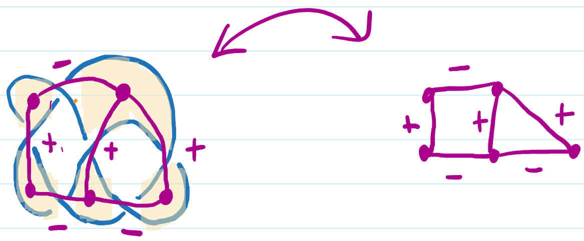
Equivalence ⟶ Reidem... Moves.

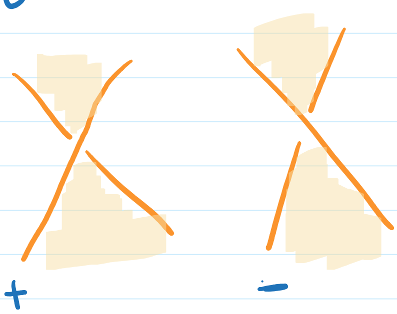
Knot Theory ⟷ Graph Theory.

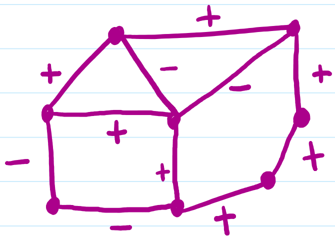
Knot proj ⟷ Graph
\+ Link.

↓

set of V, E





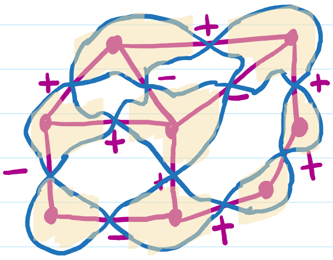


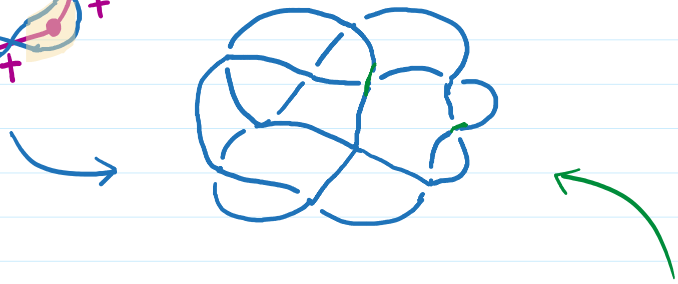
1) Is every projection mapped to a unique G?

YES

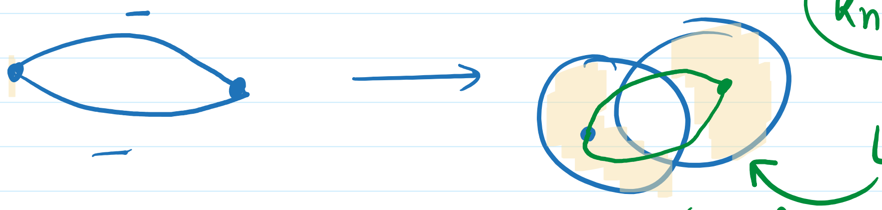
2) Taking a graph with random +, −,
Can we say it corresponds to a knot proj?

YES





Knots

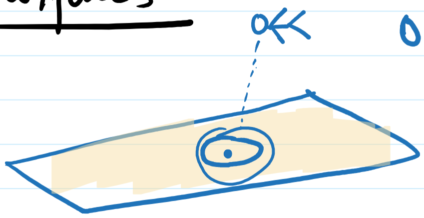


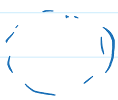
Link

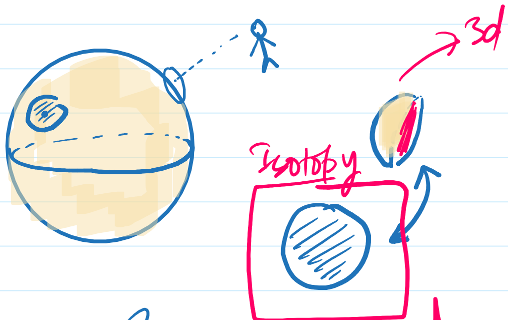
(Hopf).

## Surfaces

Open Ball







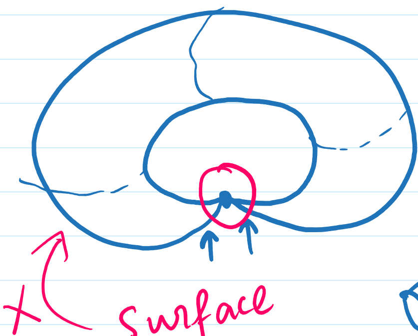
Isotopy

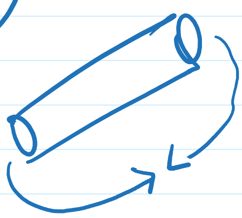
3d

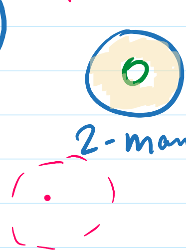
Manifolds

• open Ball - 2

2d planar





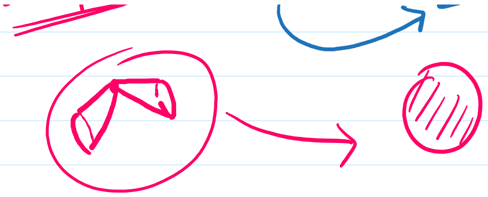


2-manifold

Surface

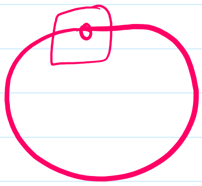
Surface

→ 2-Manifolds



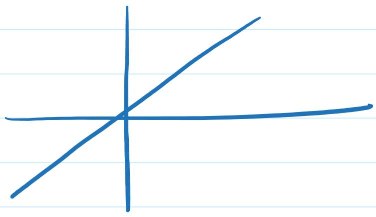
→ 2-Manifolds

1-Manifold ?



$S_1$

$R_1$



2 Manifold

3d-space

$\boxed{\text{3-Manifold}}$

3-manifold.

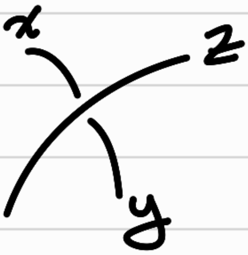
Isotopy ⟷ deformation

Surface

$\boxed{\text{4- Chapter}}$

Not

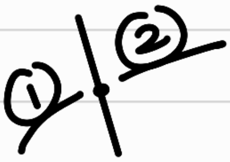
Any knot diagram is given we want to check for p-colorability.




$\underline{x+y \equiv 2z \ (p)}$

A knot with $n$ no. of strands which isn't $\underline{\text{trivial}}$. $\Rightarrow$ No. of crossings in this knot.

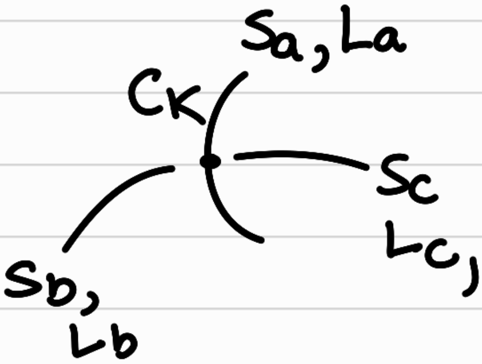
(det $\neq 0$)





$S_1 \quad S_2 \quad S_3 \; \text{-----} \; S_n$

$C_1 \quad C_2 \quad C_3 \; \text{-----} \; C_n$




$n \times n$

$$\begin{array}{c|c} & S_1 \cdots S_a - \cdots S_b - \cdots S_c - \cdots S_n \\ \begin{array}{c} c_1 \\ \vdots \\ c_k \\ \vdots \\ c_n \end{array} & \left[\begin{array}{c} \\ \\ \cdots \; \underline{\underline{2}} \;\; 0 \;\; \underline{\underline{-1}} \;\; 0 \;\; \underline{\underline{-1}} \; \cdots \\ \\ \\ \end{array}\right]_{n \times n} \end{array}$$

$M$

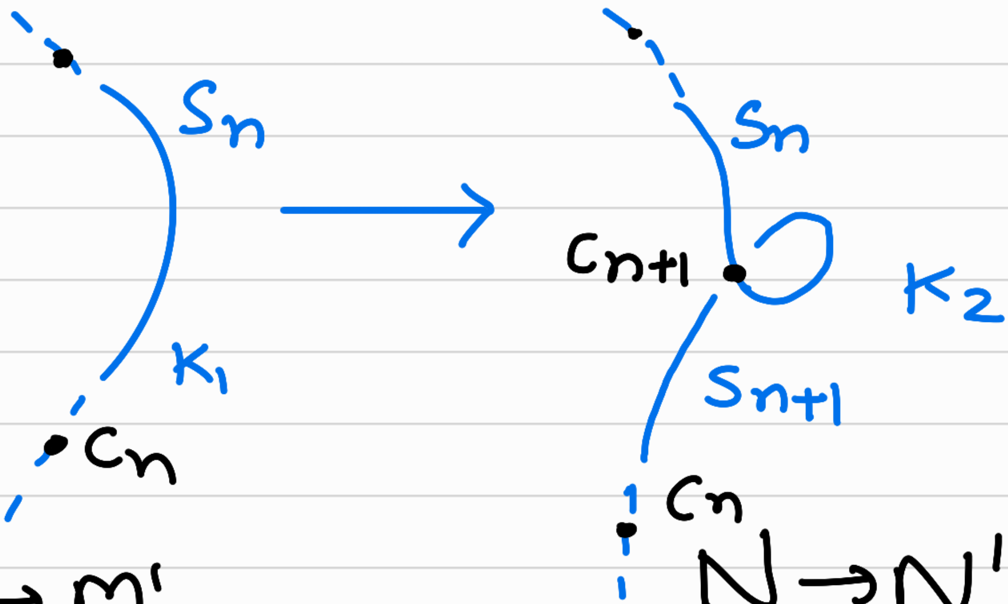
At $C_K$ for it to be p-colorable $\quad 2L_a \equiv L_b + L_c \ (p)$

$$M \begin{bmatrix} L_1 \\ L_2 \\ L_3 \\ \vdots \\ \vdots \\ L_n \end{bmatrix} = 2L_a - L_b - L_c \equiv 0 \ (p) \rightarrow k^{th} \text{ entry}$$

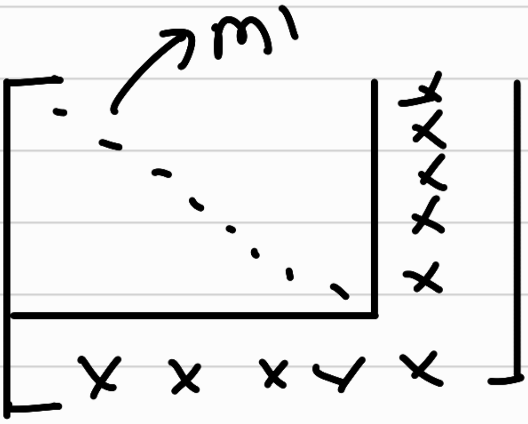
$$M \xrightarrow{\delta} (M'_{n-1 \times n-1}) = D$$

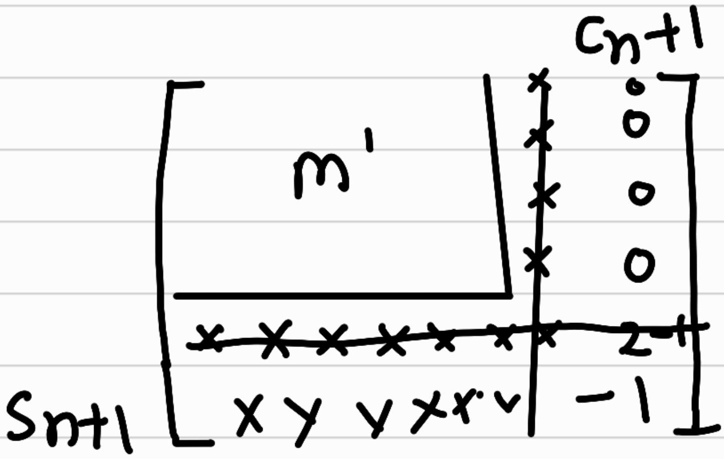
$$\underline{\det(K)} = |D|$$

$\square$ R-I




$\underset{n \times n}{M} \rightarrow \underline{\underline{M'}}$ $\qquad$ $N \rightarrow \underline{\underline{N'}}$




$$N' = \begin{bmatrix} M' & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ x\ y\ \cdots & -1 \end{bmatrix} \Rightarrow \delta(N') = -M'$$

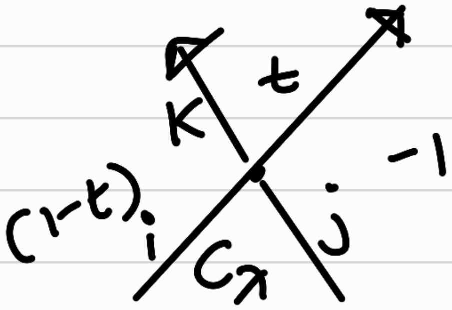
$\det(K_1) = |M'|$

$\det(K_2) = |-M'| = \det(K_1)$

$\square$ R–III (H.W.), RII (H.W.)

☆ <u>Thm.</u> : A knot diagram is p-colorable iff $\underline{\delta(M')} \equiv 0 \ (p)$

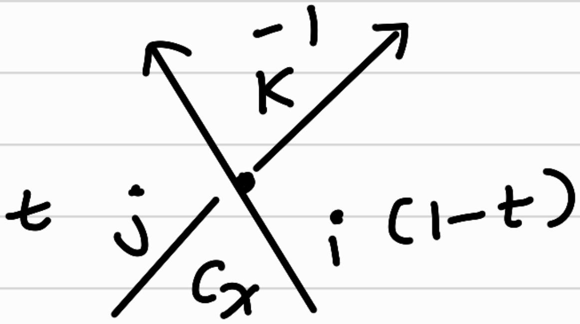
# ALEXANDER POLYNOMIAL




(a)

Right Orientation

(R)




(b)

Left Orientation

(L)

(R)

(L)

$$\begin{array}{cc} c_x & \begin{bmatrix} 0 & 0 & \underline{1-t \ \ 0 \ -1 \ \ 0 \ \ t} & 0 & \cdots \end{bmatrix} \\ & \qquad\qquad i \qquad j \qquad k \end{array}$$

$\underline{T \to T'}$

$$\begin{array}{cc} c_x & \begin{bmatrix} 0 & 0 & \underline{1-t \ \ 0 \ \ldots \ -1 \ \ldots \ 0 \ldots \ t \ldots} \end{bmatrix} \\ & \qquad\qquad i \qquad j \qquad k \end{array}$$

$T$

**Reading work:** Notations - Conway, Dowkar

Knot invariants - Bridge, Crossing Number.

Surface & Knots.

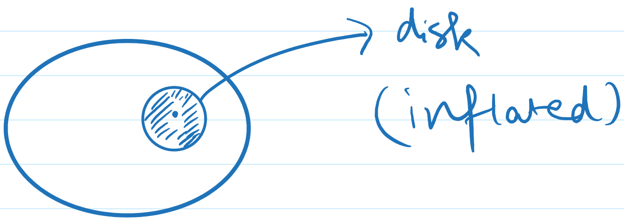
**Revision:** Surfaces, Isotrophisms.

**Lesson plans:** Triangulations, Homeomorphisms, Genus, Embedding, Euler characteristic (invariant). Compact, Knot Complement. (Surfaces with boundary), Orientable. Genus & Seifert Surfaces.

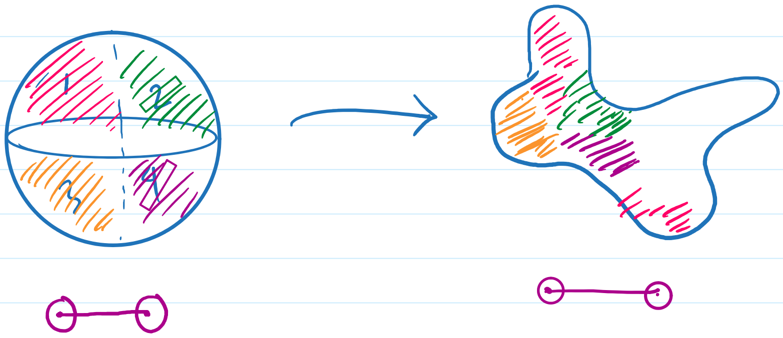
Can intro next chapter.

## Surfaces

2-manifolds. ~~1-manifolds~~




Isotopy. (deformation)



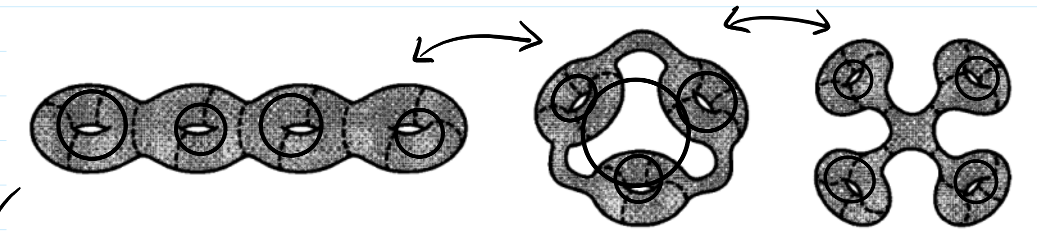



*Figure 4.7* These three surfaces are all isotopic.

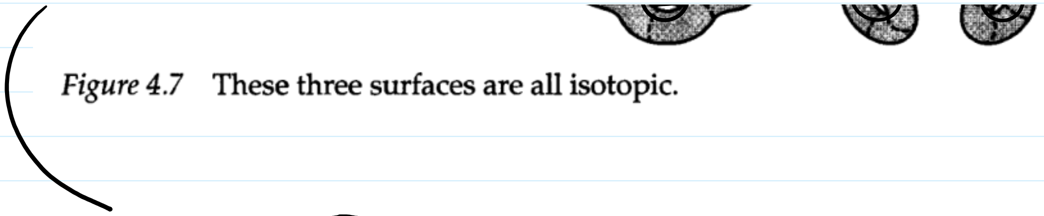
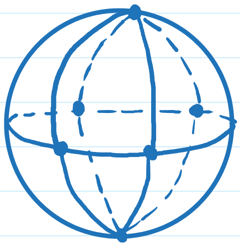


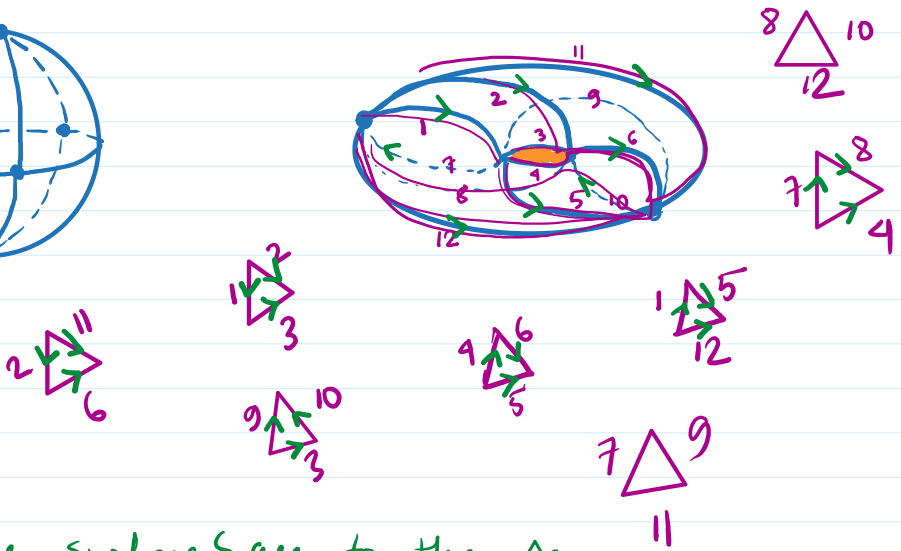
*Figure 4.7* These three surfaces are all isotopic.

not isotopic

homeomorphism

**Triangulation**: Cutting up the surface into triangles.





We cut the surface $S_1$ acc to the $\Delta$s

Glue them back acc. to the orientations & labels on the edges.

If we get another surface $S_2$ following all steps then we say $S_1$ homeomorphic to $S_2$.

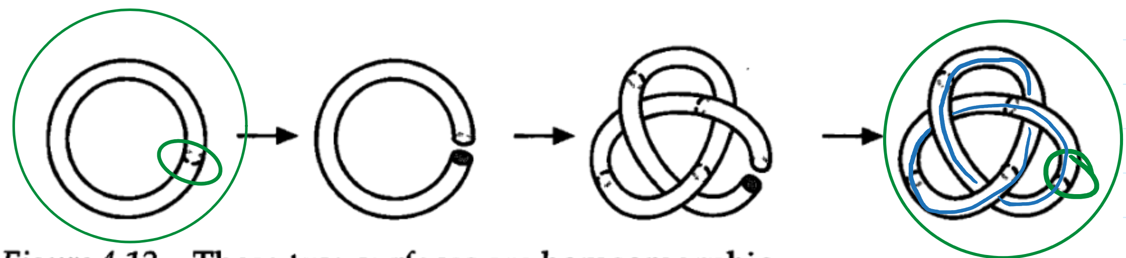
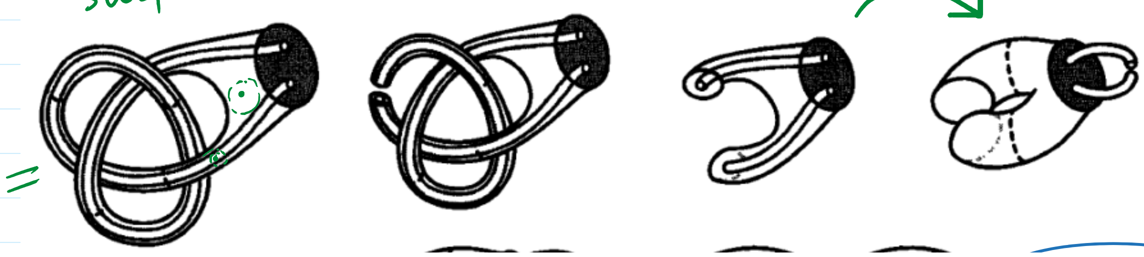


*Figure 4.12* These two surfaces are homeomorphic.

surface

#W



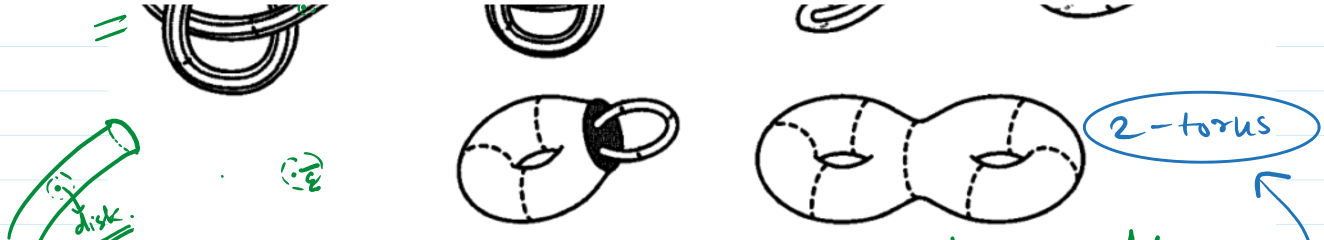
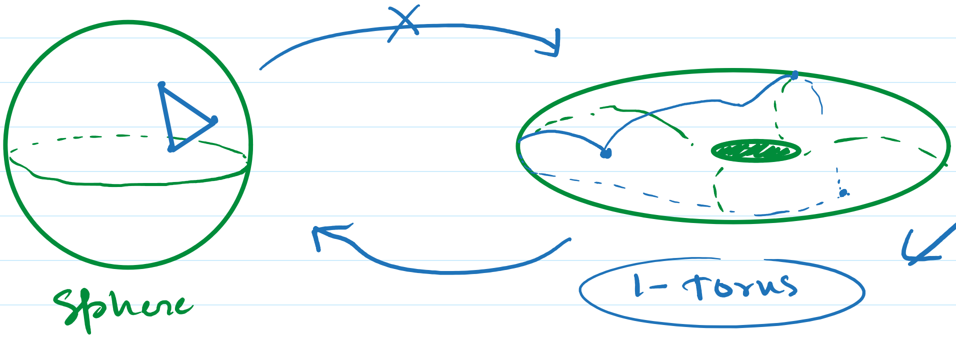


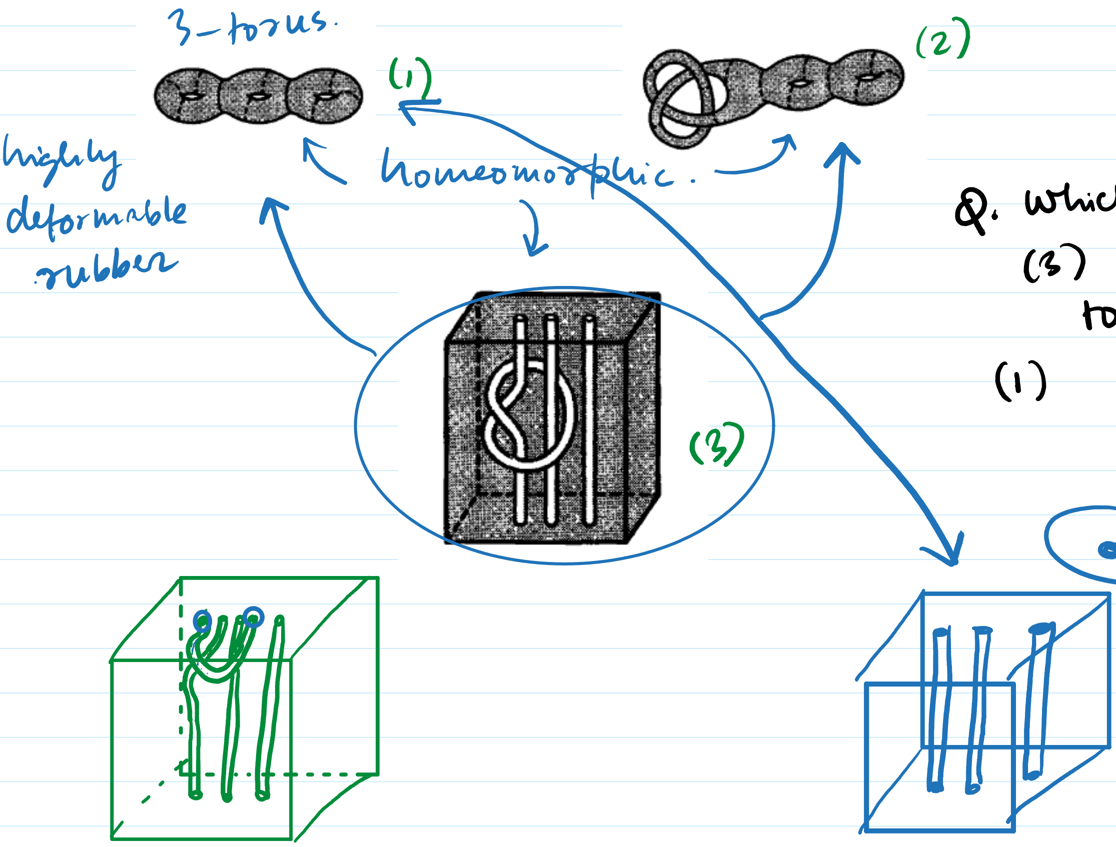

*Figure 4.13* Two homeomorphic surfaces.

(non-isotopic)



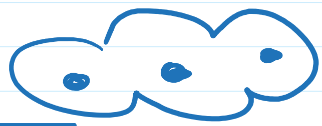

n-torus not homeomorphic to m-torus
where $n \neq m$.

<u>Genus</u>: (Generally) No. of holes in the surface.
n torus is of Genus = n.




Q. Which fig is (3) isomorphic to?

(1) (2)



embedding in a space.

**Q** Given an embedding, can we say what type of surface it is?
"Homeomorphism type"?

Sphere Torus

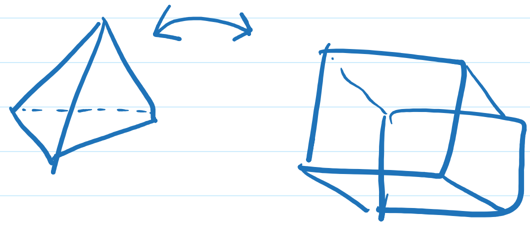
**Invariant** under homeomorphisms

$Q$ $\quad Q(S_0) = Q(\text{sphere})$

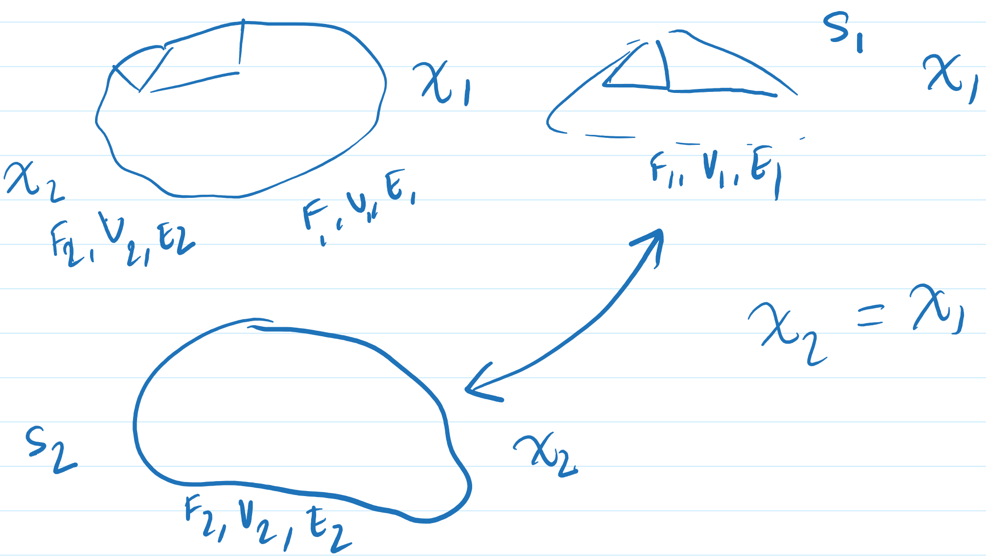
$= Q(\text{torus})$

$\underline{\chi}: V - E + F$

Euler characteristic

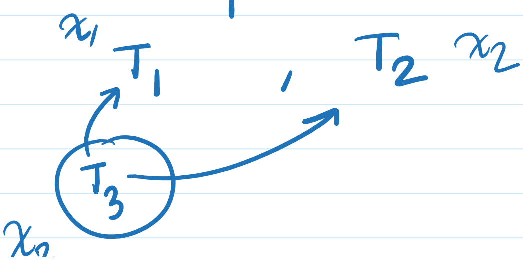


**Proof**:




$\chi_2 = \chi_1$

Surface $S$

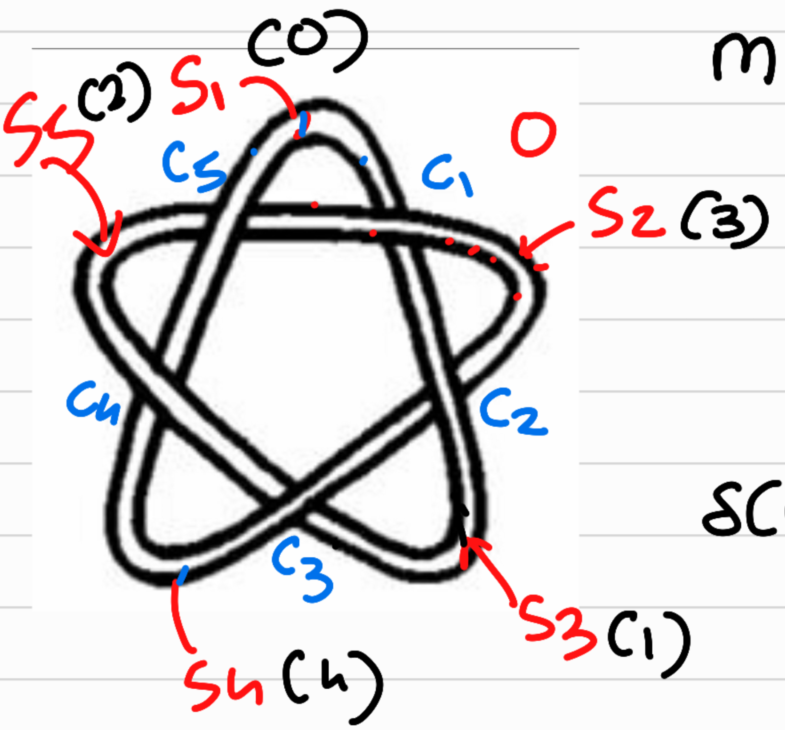



$$\begin{cases} \chi_1 = \chi_3 \\ \chi_2 = \chi_3 \end{cases}$$

$\Rightarrow \chi_1 = \chi_2$

☆ **Theorem:** For any two knot diagrams $K_1$ & $K_2$ with $T_1'$ & $T_2'$

The $\delta(T_1') = \pm t^k \delta(T_2')$
where $k \in \mathbb{Z}$




$$m = \begin{bmatrix} -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \\ \cancel{2} & \cancel{-1} & \cancel{0} & \cancel{0} & \cancel{-1} \end{bmatrix}$$

$$\delta(m) = \begin{vmatrix} -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \\ -1 & 0 & 0 & -1 \end{vmatrix}$$

$$= -1 \times \begin{vmatrix} -1 & 2 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{vmatrix} - (-1) \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix}$$

$$= 1 + 1\left[2(3) + 1(-2)\right]$$

$$= 1 + 4 = \underline{\underline{5}}$$

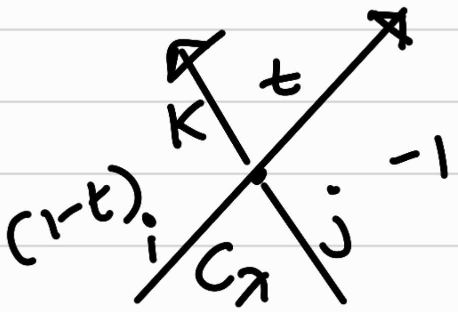
**Q.** Which of the following is not a possible $\delta$.

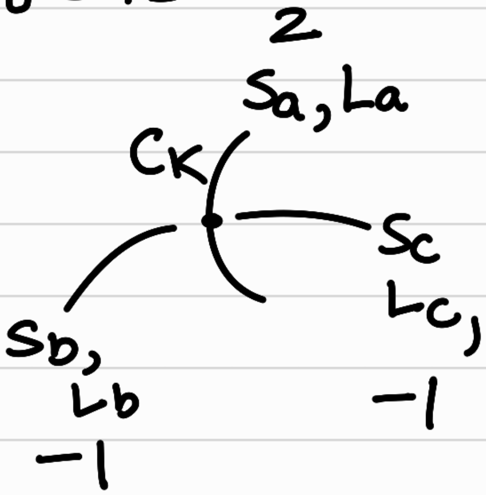
a) 15 b) 97 c) 34 d) 121

• **Mod p-rank of a Knot:-** It is basically the nullity of the $(n-1)\times(n-1)$ matrix.

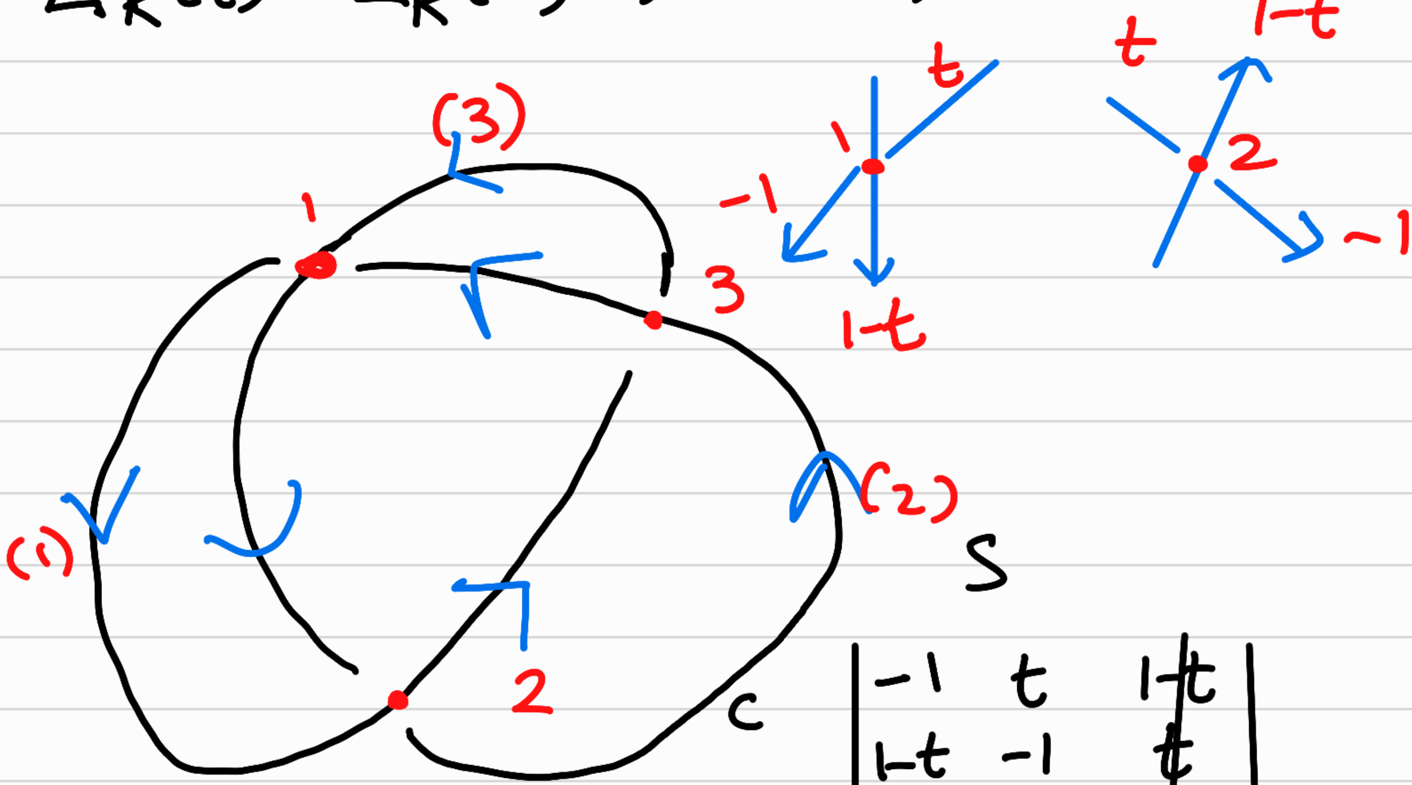
$\Uparrow$
Invariant (R-moves)

$8_{18}$ & $9_{24}$ $\Rightarrow \det = 45$



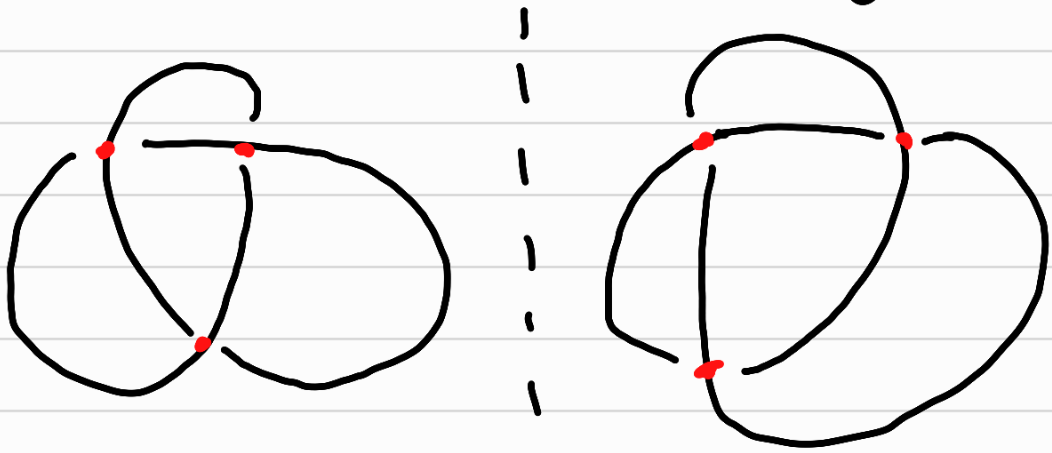

$$\Delta_K(t) = \Delta_K(-1) \Rightarrow \det(K)$$

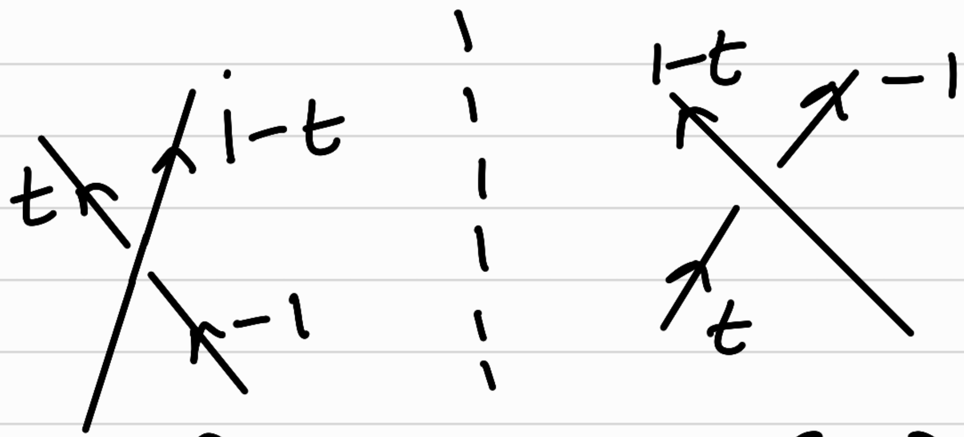



$$\begin{vmatrix} -1 & t & \cancel{1-t} \\ 1-t & -1 & \cancel{t} \\ \cancel{t} & \cancel{1-t} & \cancel{-1} \end{vmatrix}$$

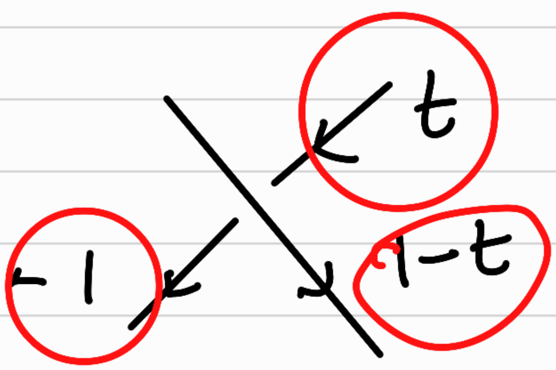
$$\Delta_{\text{Trefoil}}(t) = \underline{t^2 - t + 1} = t\underbrace{\left(t - 1 + \frac{1}{t}\right)}$$

# Drawback of alexander polynomial :



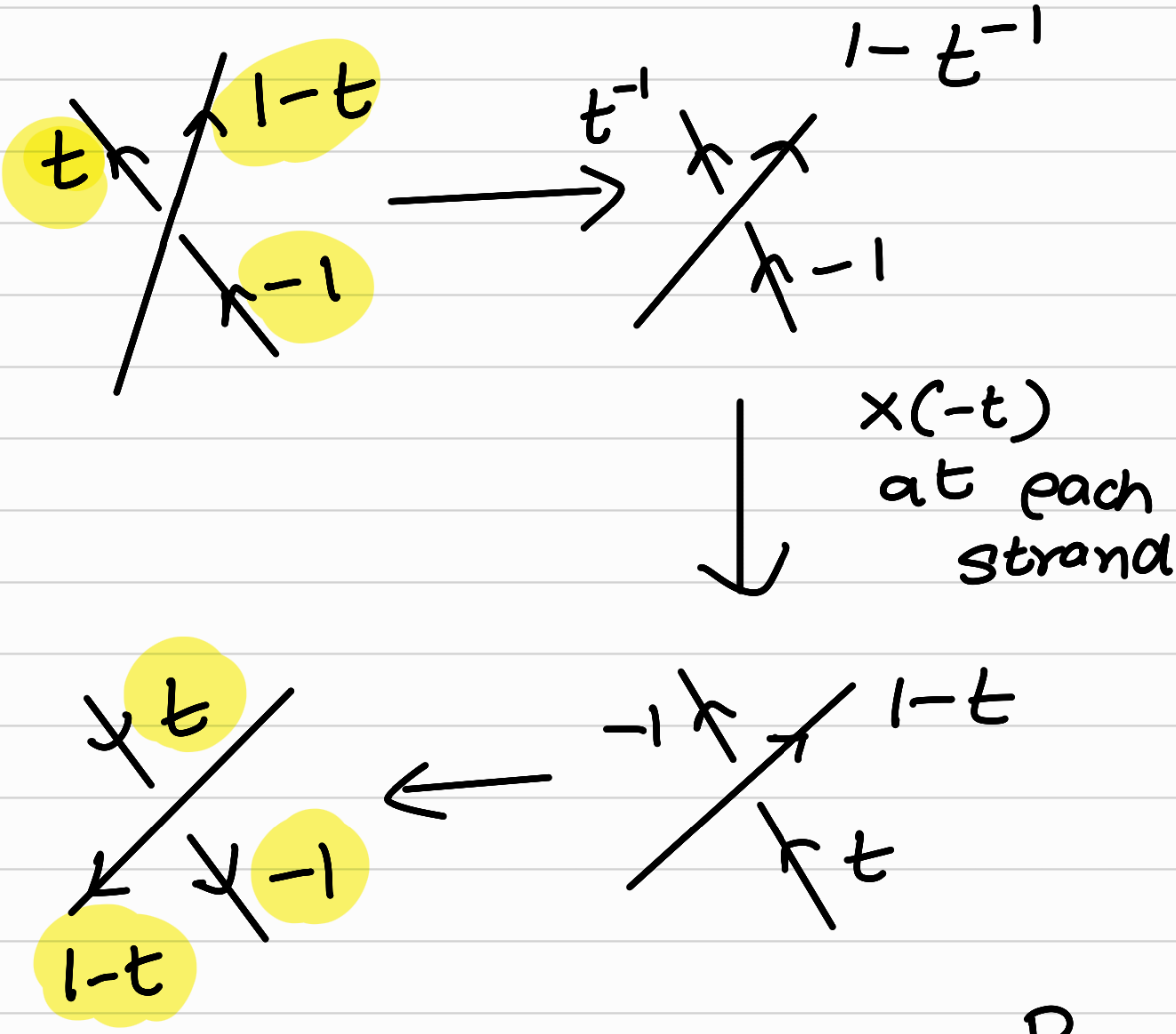
• 


(R) (L)




$\Rightarrow \Delta_K(t) = \Delta_{\bar{K}}(t)$

• **Prop$^n$**: $\Delta_K(t) = \Delta_K(t^{-1})$




$$\Delta_K(t) = \pm t^k \left( \begin{array}{l} \text{A symmetric} \\ \text{poly. b/w } t \\ \text{and } 1/t \end{array} \right)$$

P

basically coeff. of $t^n$ = coeff. of $t^{-n}$.
(in P)

# G.T. Revision

* **Group:-** Closed, identity, associative, inverse

* **Subgroup:-** $H \subseteq G$
  and $H$ is a gp.
  $H \leq G$

* **Group Homo:-**

$$f(a \cdot b) = f(a) \cdot f(b)$$

* **Iso.** * **Auto.**

* **Homo.** (Kernel)

* **Normal sub. gp.**

$H \leq G$

$gHg^{-1} = H \ \forall g \in G$

$\Rightarrow H \trianglelefteq G$

* $K = \ker(G)$

$K \trianglelefteq G \quad (\varphi)$

$K' = \{ \underline{gkg^{-1}} \mid k \in K \}$

$\varphi : G \longrightarrow \tilde{G}$

$K := \ker_{\varphi}(G)$

$gkg^{-1} \in K$

$$\varphi(gkg^{-1}) = \varphi(g)\,\underbrace{\varphi(k)}_{\tilde{e}}\,\varphi(g^{-1})$$

$$= \varphi(g) \cdot \varphi(g^{-1})$$

$$= \varphi(gg^{-1}) = \varphi(e) = \tilde{e}$$

$\Rightarrow gkg^{-1} \in K$

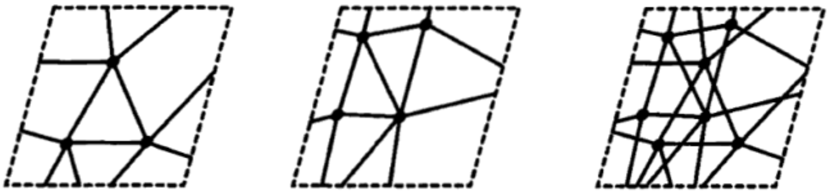
For any $k \in K$ $\exists$ $k'$ s.t.

$gk'g^{-1} = k$

$k' = g^{-1}kg \in K$

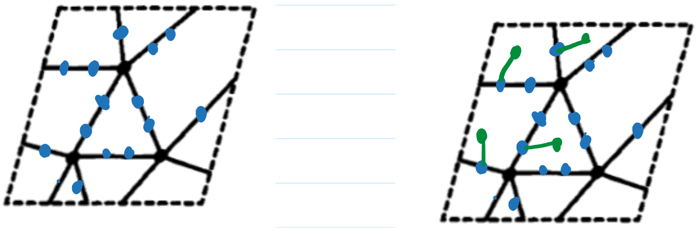
$\chi_3$ $T_3$

$\Rightarrow \chi_1 = \chi_2$

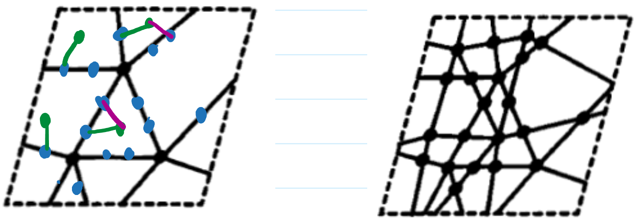


a $= T_1$ b $= T_2$ c $= T_1 \cup T_2$

$\chi : \underset{+}{V} - \underset{+}{E} + F$



a a



a b

$+E$
$+F$ $\chi$

## Steps to Construct $T_3$ :-

1) Start with $T_1$

2) Mark the vertices where the edges of $T_2$ intersect $T_1$

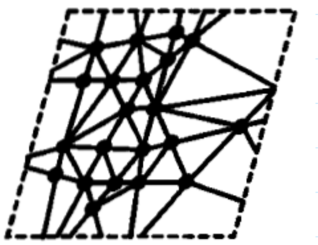
$\chi$ is constant as $V+$, $E+$.

3) Mark the vertices of $T_2$ and join them with edges to the intersection points.

and join them with edges to the intersection points.
$V+$ , $E+$

4) Join the remaining $T_2$ edges in $T_3$
Even here $\chi$ is constant as $E+$ $F+$

5) Join the vertices to form triangles
$\chi$ is constant as $E+$ $F+$.



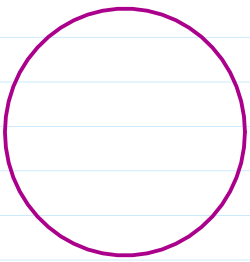
$c = T_1 \cup T_2$

$$\chi(c) = \chi(T_1)$$
$$\chi(c) = \chi(T_2)$$
$$\Rightarrow \chi(T_1) = \chi(T_2)$$

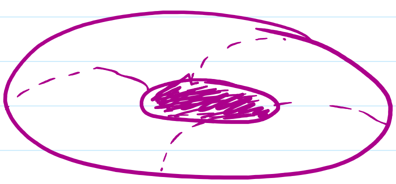
$\chi(\text{sphere}) = \underline{\underline{x}}$

$\chi(\text{torus}) = y$.

$\chi = z$



$\chi(\text{sphere}) = 2$
(0-torus)



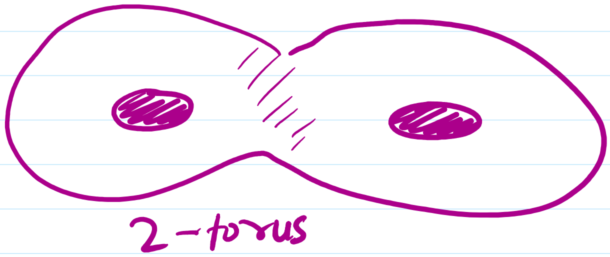
$\chi(\text{torus}) = 0$
(1-torus)
$V = 4, \quad E = 12, \quad F = 8$

$$\boxed{\chi = 4 - 12 + 8 = 0}$$

torus $\rightarrow$ tori

H/W

H/W



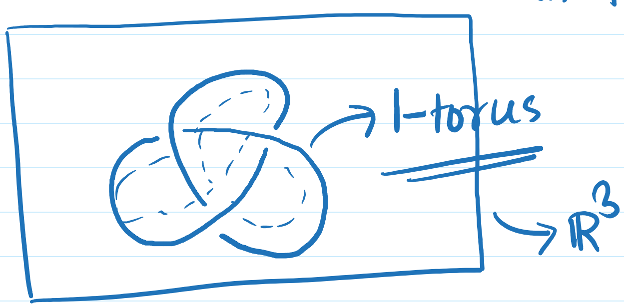

$\chi$ (2-torus) $= -2$

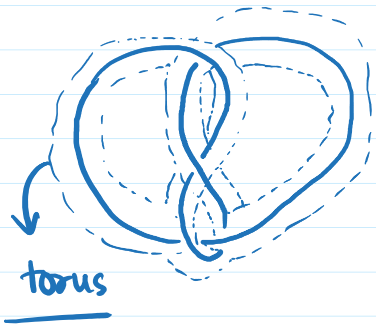
$$\boxed{\chi(n\text{-torus}) = 2 - 2n}$$

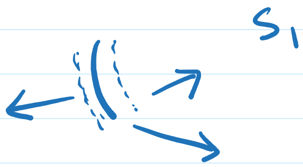
Genus $\longleftrightarrow$ $\chi(\dots)$

Knot $\longrightarrow$ Embedding of $S^1$ onto $\mathbb{R}^3$.

Knot Complement :- Basically everything except the knot. in the space $\mathbb{R}^3$



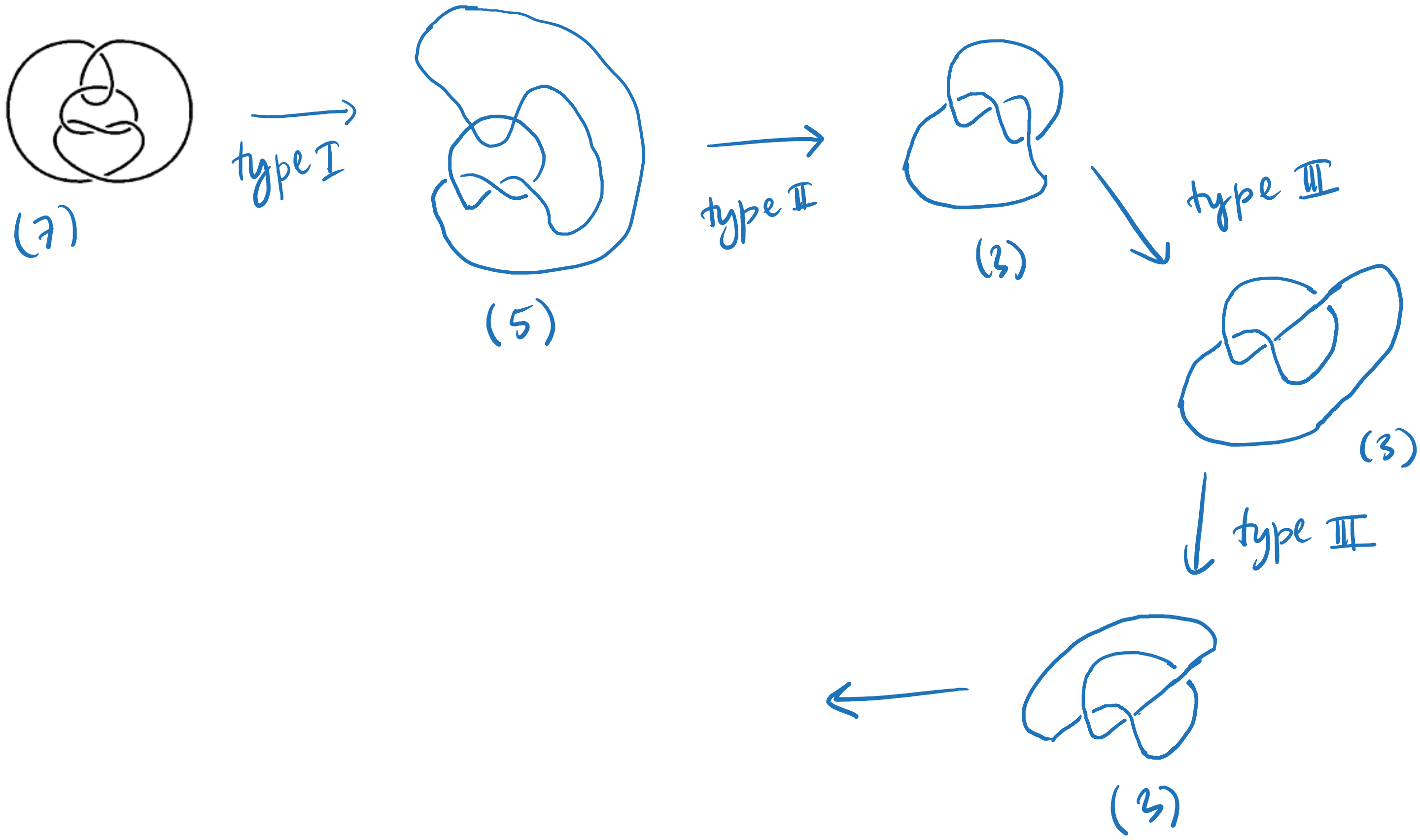




" All the surfaces live in the complement of the knots. " links.

Surfaces without boundary.

# Exercises

Friday, 22 November, 2024 12:03 PM



# Permutations

$P$ is a permutation on a set $S$.

$\Rightarrow P(s_1) = P(s_2)$ iff $s_1 = s_2$

$\Rightarrow P : S \longrightarrow S$

$\Rightarrow \forall s \in S \ \exists! s_0 \in S$ s.t. $P(s_0) = s$

- An iso. is an example of a permutation.
- $[n] = \{1, 2, ---, n\} / 1(1)n$

$S_n := \{ P \mid P$ is a permutation from $[n]$ to $[n] \}$

- A group
- Identity is just $x \mapsto x \ \forall x \in 1(1)n$
- $\# |S_n| = n!$

- generator sets :-

The entire group can be generated by the generators and their inverses.

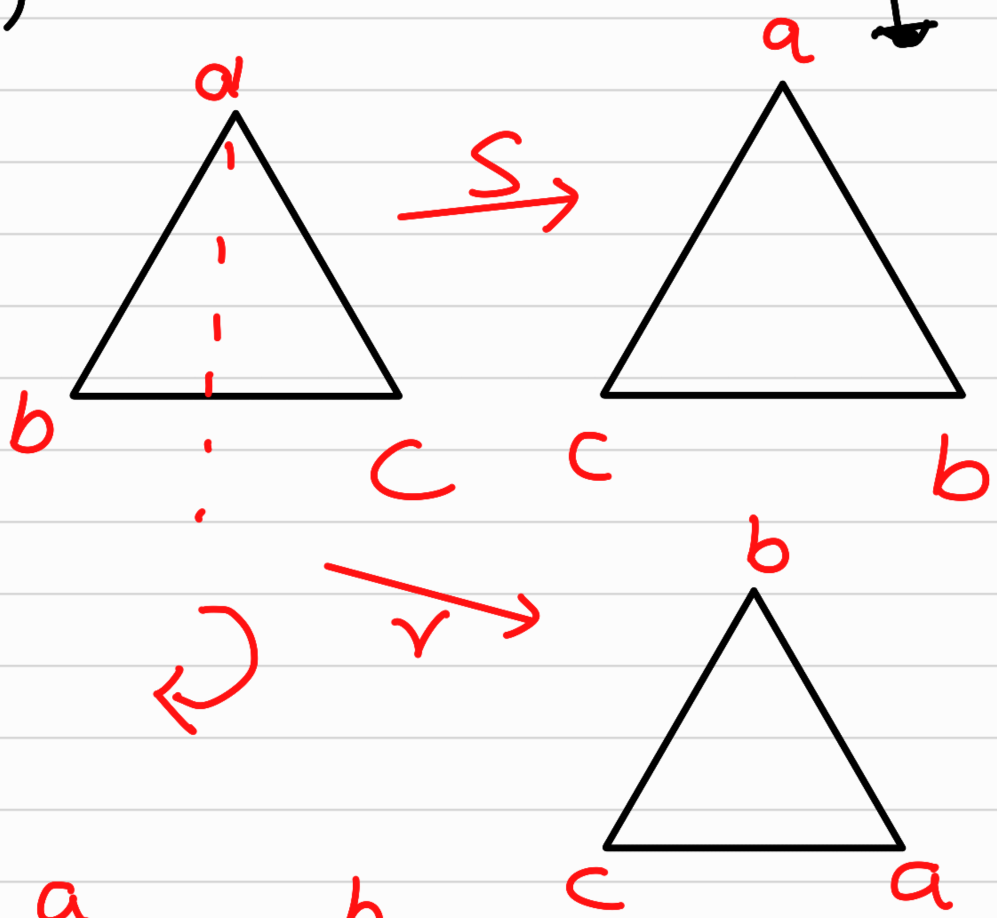
# Dihedral group

$D_n \Rightarrow \{s, r\}$ $\boxed{n \geq 3}$

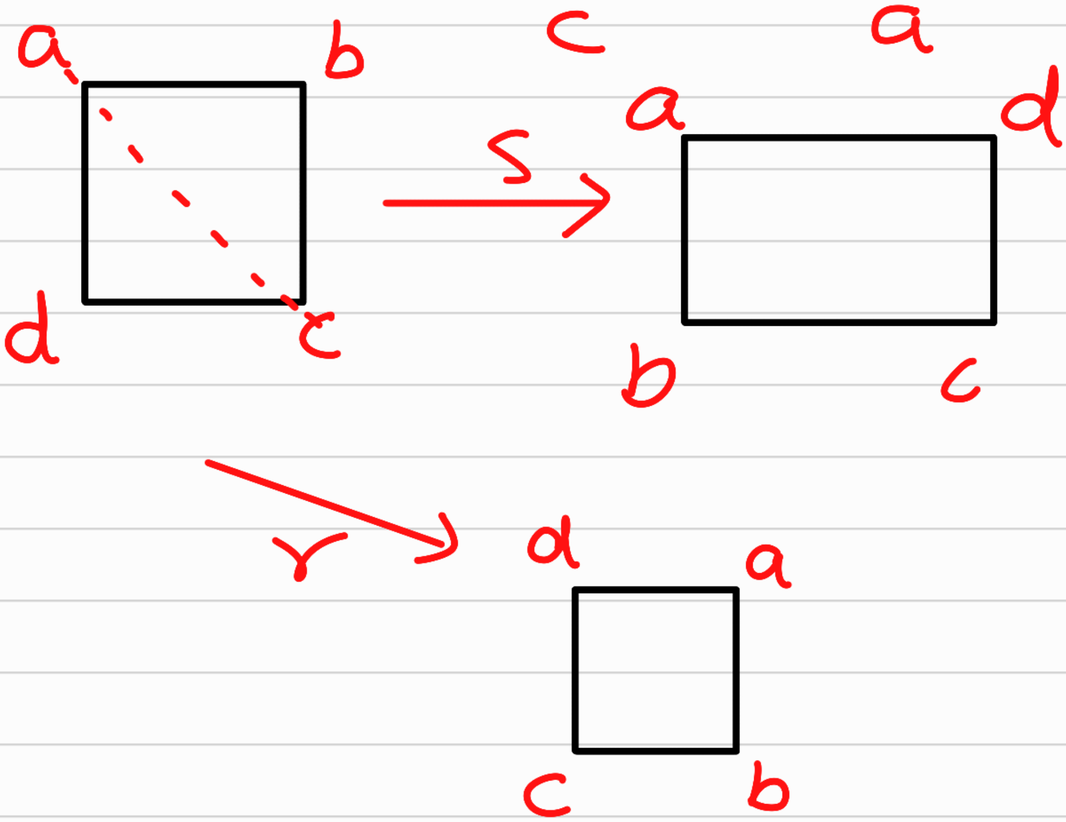
(Regular Polygon of size n)

$\underline{s^2 = e}$, $\underline{r^n = e}$, $\underline{(sr)^2 = e}$

$D_3$




$D_4$



$srsr = e \Rightarrow \underline{srs} = \underline{r^{-1}}$

**H.W.**: $sr^k s = r^{n-k}$

# Cyclic Notation.

$P: [5] \longrightarrow [5]$

$1 \to 4$
$2 \to 3$
$3 \to 1$
$4 \to 2$
$5 \to 5$

$P = (1\,4\,2\,3)\ \underline{(5)}$

# Thm.: Any permutation can be represented as product of cycles with empty intersection.

**Proof:** Say we are working on a set $S$ s.t. $|S| = n$

$a_1 \in S$

$$a_1, P(a_1), P(P(a_1)) \ldots, \underset{\substack{\shortparallel \\ p^{\ell}(a_1)}}{P^k(a_1)}$$

where $\ell \neq 0$ & All the elements

before $p^k$ were distinct.

$$p(\underline{p^{l-1}(a_1)}) = p^{l}(a_1)$$

$$p(\underline{p^{k-1}(a_1)}) = p^{k}(a_1) = p^{l}(a_1)$$

$\Rightarrow p^k$ will terminate at $a_1$.

$(a_1, p(a_1), \ldots, p^{k-1}(a_1)) \rightarrow k \geq 1$

$b_1$ . - - - - $\qquad l \geq 1$

$c_1$ - - - - $\qquad m \geq 1$

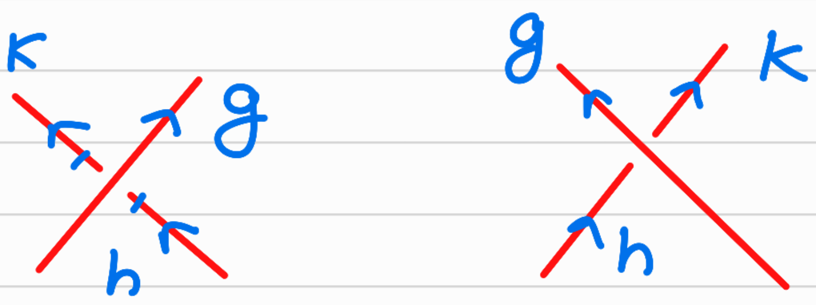
- Transposition: A 2 cycle.

☆ Thm.
Any permutation can be written as a product of transpositions.

# Knots & Groups

• Group is $G$ & any knot is $K$.

*




(Right) (Left)

$gkg^{-1}=h$ $ghg^{-1}=k$

OR $gk=hg$ OR $gh=kg$

☆ All the assigned labels at the crossing end up generating $G$.

Q. If we know $k$ and $h$ does our $g$ get fixed?

$\Rightarrow k=h=e \quad \forall g\in G$

# Thm. If any K. diag. can be labeled with elements from G, then any of the K. diag. of the same knot can be labeled with elements from G, regardless of whatever orientation choice.

$\Rightarrow$ R–I, R–II, R–III.

Conway $\longrightarrow$

$\boxed{\underline{t_1}, \underline{t_2}}$