Mathematics Beyond Precision: Navigating the World of (Fuzzy) Uncertainty

Daibik Barik

Math Club, ISI Bangalore

February 8, 2025

Outline

What is Fuzzy?

Sentences like

- *Today's temperature is hot* or
- *The cake is sweet*

are used in our everyday conversation.

Think!

Are they mathematically rigorous?

Well, even the question wasn't clear.

Why Fuzzy?

Question

I thought we used probability to deal with uncertainties, why don't we use that here?

Short answer: Probability deals with *Stochastic ambiguities*, where an experiment is performed repeatedly and

$$\text{probability} := \frac{\text{No. of favourable outcomes}}{\text{No. of total outcomes}}$$

Why Fuzzy?

That doesn't happen in this case. Here the problem deals with the definition being *inherently vague*, whose *meaning changes* from person to person.
This can be further complicated with the use of modifiers like "very", "quite", "not", "somewhat", "too" etc.

Question

Is "not tall" ≡ "short"? Or "not quite tall" ≡ "short"?

Hint: Try to negate the statements.

How do we define?

Well, *Zadeh(1965)* did that for us.
He gave the concept of **Membership of an element in the set**

Membership (μ)

(Generalising Crisp Sets) Instead of an element belonging to a set, *Zadeh* says it has a degree of membership $\in [0, 1]$.
In a Crisp set A, membership of x is denoted by

$$\mu_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Generalization

Crisp Sets

- **Binary Membership:** An element either belongs to the set ($\mu(x) = 1$) or does not ($\mu(x) = 0$).
- **Logical Operations:** Uses classical logic (AND, OR, NOT).
- **Example:** The people taller than 6 feet.

Fuzzy Sets

- **Gradual Membership:** An element can partially belong to a set with a $\mu(x)$ having value between 0 and 1.
- **Fuzzy Operations:** Uses fuzzy logic (e.g., fuzzy union, intersection, complement).
- **Example:** Tall people.

Examples (Discrete)

Let $A =$ About 10 feet.
Surely $10 \in A$.

Question

What about 8, 9,or say, 11, 12?

We assign Membership to them, say $9, 11$ have 0.9 membership, $8, 12$ have 0.7 membership and 7 has 0.5 membership
A can be represented as

$$A = \left\{ \frac{1}{10} + \frac{0.9}{9} + \frac{0.9}{11} + \frac{0.7}{12} + \frac{0.7}{8} + \frac{0.5}{13} + \frac{0.5}{7} \right\}$$

Examples (Continuous)

Let $A =$ About 10 feet.

Question

What about values like 9.8 or 10.7? Can it include all the neighborhood points in a continuous sense in $\mathbb{R}$?

So we define $\mu_A : A \to [0,1]$ where A is a interval containing 10, say $A = [7,13]$. We can also generalize to $\mathbb{R}$ by assigning membership 0 at other points.

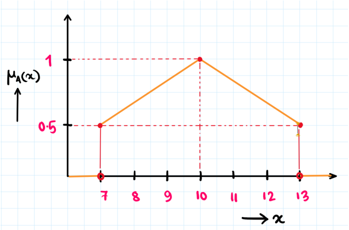

Figure: The $\mu_A(x)$ graph

$$\mu_A(x) = \begin{cases} 0 & \text{if } x \in (-\infty, 7) \\ \frac{1}{6}x - \frac{2}{3} & \text{if } x \in [7, 10) \\ -\frac{1}{6}x + \frac{8}{3} & \text{if } x \in [10, 13] \\ 0 & \text{if } x \in (13, \infty) \end{cases}$$

Note: that μ_A is continuous at $x = 10$.

Set operations

We intend to define Union, intersection and complement of a Fuzzy Set using the membership function.

Characteristic Function (χ)

Temporarily, for crisp sets, let's construct a membership (characteristic) function χ which maps the elements to $\{0, 1\}$

Now the rules followed by Crisp sets will lay down the ground rules and we try to generalize them for fuzzy sets.

Complementation

Say $U = \{1, 2, 3, \ldots, 6\}$
Let $\exists$ a crisp set, say $A \subseteq U$.

$$A = \{2, 4, 5\}$$

$$\implies \overline{A} = \{1, 3, 6\}$$

Note:

$\chi_A(1) = 0,\ \chi_{\bar{A}}(1) = 1$
$\chi_A(2) = 1,\ \chi_{\bar{A}}(2) = 0$

$$\boxed{\therefore \overline{A} \equiv \textbf{NOT } A}$$

A complementation function may be denoted as $c : [0,1] \rightarrow [0,1]$ such that if $\mu_A(x) = a \in [0,1]$ then $\mu_{\bar{A}}(x) = c(a)$.

Essential Properties:-

- *(Boundary Condition)* $c(1) = 0 \quad \& \quad c(1) = 0$.
- *(Monotonicity)* If $a \leq b$, then $c(a) \geq c(b)$.

Desired Properties:-

- c is continuous.
- c is involutive i.e. $c(c(a)) = a$.

Note: The properties are not independent under some theorems.

Theorem

The following hold:-

(a) *If c is involutive, then c is bijective.*

(b) *If c is monotonic and involutive, then $c(0) = 1$ and $c(1) = 0$.*

(c) *If c is bijective, then it is continuous.*

Theorem

A complementation function has **atmost** *1 equilibrium point.*

From sign changing property of continous functions, we can say they have **atleast** 1 equilibrium point.
$\implies$ Continious complementation functions have exactly 1 equilibrium point.

Complementation Examples

Let A be a fuzzy set and x be having a membership of $\mu_A(x)$ in A. The different types of complementation functions that satisfy the conditions are:-

1. Standard Complement $\rightarrow$ $c(\mu_A(x)) = 1 - \mu_A(x)$
2. Sugeno Complement $\rightarrow$ $c_\lambda(\mu_A(x)) = \frac{1-\mu_A(x)}{1+\lambda\mu_A(x)}$
3. Yager Complement $\rightarrow$ $c_w(\mu_A(x)) = (1 - \mu_A(x)^w)^{1/w}$

Example: (Alternate Representation)

$U = \{1, 2, 3, \ldots, 6\}$ & $A = \{(2, 0.5), (5, 0.2), (6, 1)\}$
Standard Complement
$\rightarrow \overline{A} = \{(1, 1), (2, 0.5), (3, 1), (4, 1), (5, 0.8)\}$

Union & Intersection

Say $U = \{1, 2, 3, \ldots, 10\}$.
Let $\exists$ two crisp sets, $A, B \subseteq U$:

$$A = \{2, 4, 6, 7\}, \quad B = \{1, 3, 6, 7\}.$$

Then:

$$A \cup B = \{1, 2, 3, 4, 6, 7\}, \quad A \cap B = \{6, 7\}.$$

Note:

$$\chi_A(1) = 0,\ \chi_B(1) = 1 \implies \chi_{A\cup B}(1) = 1,\ \chi_{A\cap B}(1) = 0$$
$$\chi_A(2) = 1,\ \chi_B(2) = 0 \implies \chi_{A\cup B}(2) = 1,\ \chi_{A\cap B}(2) = 0$$
$$\chi_A(6) = 1,\ \chi_B(6) = 1 \implies \chi_{A\cup B}(6) = 1,\ \chi_{A\cap B}(6) = 1$$
$$\chi_A(9) = 0,\ \chi_B(9) = 0 \implies \chi_{A\cup B}(9) = 0,\ \chi_{A\cap B}(9) = 0$$

$$\boxed{\therefore \bigcup \equiv \textbf{OR}, \bigcap \equiv \textbf{AND}}$$

A union (or intersection) function may be denoted as $u : [0,1] \times [0,1] \to [0,1]$ such that if $\mu_A(x) = a,\ \mu_B(x) = b$ and $a, b \in [0,1]$ then $\mu_{A \cup B}(x) = u(a,b)$.

Essential Properties:-

- *(Boundary Condition)* $u(a,0) = a$.
- *(Monotonicity)* If $b \leq d$, then $u(a,b) \leq u(a,d)$.
- *(Commutativity)* $u(a,b) = u(b,a)$
- *(Associativity)* $u(a,u(b,d)) = u(u(a,b),d)$

Desired Properties:-

- u is continuous.
- $u(a,a) \geq a\ [\because 0 \leq a \implies u(a,0) = a \leq u(a,a)]$
- If $a_1 < a_2$ & $b_1 < b_2$ then $u(a_1,b_1) < u(a_2,b_2)$

Note: We can similarly construct a function $i : [0,1] \times [0,1] \to [0,1]$ for intersection and observe the properties.

Idempotency

A property of an operation that can be repeated multiple times without changing the outcome. Here it means $f(a,a) = a$. So 'unions' are super-idempotent, and 'intersections' are sub-idempotent.

Theorem

The standard union, i.e. $u(a,b) = max\{a,b\}$ and the standard intersection, i.e. $i(a,b) = min\{a,b\}$ are the only idempotent functions satisfying the conditions.

Examples

Let A, B are two fuzzy sets and x be having a membership of $\mu_A(x)$ in A.
The different types of union (and intersection) functions that satisfy the conditions are:-

1. Standard $\rightarrow$ $u_m(a, b) = \max\{a, b\}$
   $i_m(a, b) = \min\{a, b\}$
2. Algebraic Sum $\rightarrow$ $u_a(a, b) = a + b - ab$
   $i_a(a, b) = ab$
3. Bounded Sum $\rightarrow$ $u_b(a, b) = \min\{1, a + b\}$
   $i_b(a, b) = \max\{0, a + b - 1\}$
4. Drastic $\rightarrow$

$$u_d(a, b) = \begin{cases} a & \text{if } b = 0 \\ b & \text{if } a = 0 \\ 1 & \text{otherwise} \end{cases} \qquad i_d(a, b) = \begin{cases} a & \text{if } b = 1 \\ b & \text{if } a = 1 \\ 0 & \text{otherwise} \end{cases}$$

Standard in use

Example: (Alternate Representation)

$U = \{1, 2, 3, \ldots, 6\} \quad \&$
$A = \{(1, 0.5), (2, 0.2), (4, 1), (6, 0.7)\}$
$B = \{(1, 0.3), (3, 0.5), (5, 0.2), (6, 1)\}$

Standard Union
$\rightarrow A \cup B = \{(1, 0.5), (2, 0.2), (3, 0.5), (4, 1), (5, 0.2), (6, 1)\}$
Standard Intersection
$\rightarrow A \cap B = \{(1, 0.3), (6, 0.7)\}$

Set Properties

The properties which fuzzy sets follow are:-

- **Scalar cardinality** $\rightarrow$ Defined as $|A| := \sum_x \mu_A(x)$
- **De Morgan's Law** $\rightarrow$ Both the equations involve complements A^c, unions ($\bigcup$) and intersections ($\bigcap$).
- **Subsethood** $\rightarrow$ A contains B if $\forall\ x \in U,\ \mu_B(x) \leq \mu_A(x)$.
  If the set B is not entirely a subset of A, then we can define the *degree of subsethood of B in A.*

But there are properties which the Fuzzy sets don't follow. Like:-

- **Excluded Middle**$\rightarrow$ i.e. $A \cup A^c = U$
- **Contradiction** $\rightarrow$ i.e. $A \cap A^c = \phi$

When I say don't follow, I mean the standard definition triplet of fuzzy union, intersection and complementation, i.e., $\left\langle \max\{\mu_A, \mu_B\}, \min\{\mu_A, \mu_B\}, 1 - \mu_A \right\rangle$.
But there exists triplets which follow this, but then the De Morgan's laws are not satisfied.

De Morgan's Laws $\rightarrow$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$;
$\overline{A \cap B} = \overline{A} \cup \overline{B}$

What numbers?

The numbers like *near* 100, *about* 20 are fuzzy numbers, as well as the numbers like 46 or 254. The mathematical definition was, again, given by Zadeh.

But we need some basic definitions before we can rigorously proceed towards it.

Definitions

Let $A = \left\{ \frac{1}{1}, \frac{0.5}{2}, \frac{0.7}{3}, \frac{0.9}{8} \right\}$ with membership function μ_A.

1. **Normality:**
   A set A is said to be normal if $\exists\ x \in A$ such that $\mu_A(x) = 1$.
   $\rightarrow$ As $\exists\ 1 \in A$ with $\mu_A(1) = 1$, A is *normal*.
2. **Support:**
   The support of a fuzzy set A is the crisp subset of all elements in Universal set, that have a membership grade > 0 in A.
   $\rightarrow\ suppA = \{1, 2, 3, 8\}$

3 Alpha (α) cuts:

Let there be an $\alpha \in [0, 1]$
An alpha(α)-cut is a crisp set having elements of A with membership $\geq \alpha$, i.e.

$$\alpha_A := \{x \mid \mu_A(x) \geq \alpha\}$$

$$\rightarrow \quad \alpha\text{-cut} = [x_\alpha, y_\alpha]$$

$$\frac{a - x_\alpha}{a - b} = \frac{\alpha}{1}$$

$$\implies \quad a - x_\alpha = \alpha(a - b)$$

$$\implies \quad x_\alpha = a + \alpha(b - a)$$

Let A comprises of $x \in [a, b]$ with membership function denoted as

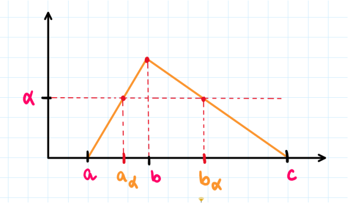

Figure: α-cut of $[a, b]$

Fuzzy Numbers

Definition

A fuzzy number A is a fuzzy set defined over $\mathbb{R}$ with the following properties:-

- A is normal, i.e., $\exists\ x \in A$ such that $\mu_A(x) = 1$.
  (*about* 20 should contain 20)
- Support(O_A^+) of A should be bounded,
  i.e., $\exists\ w_1, w_2$ such that $\mu_A(x) = 0\ \forall x \in (-\infty, w_1) \cup (w_2, \infty)$.
- All α-cuts of A should be bounded,
  i.e., $\alpha_A = [x_\alpha, y_\alpha]\ \forall \alpha \in [0, 1]$.

But how?

How to check that all the α-cuts are closed intervals?
Do we need a mathematical function?

Theorem

(Fuzzy Numbers) Let A be a fuzzy number. Then $\mu_A(x)$ can be characterized as follows.

$$\mu_A(x) = \begin{cases} 1 & \textit{on some interval } [a, b] \\ l(x) & \textit{if } x \in (-\infty, a) \\ r(x) & \textit{if } x \in (b, \infty) \end{cases}$$

where:

- $l(x)$ *is a monotonically non-decreasing function in* $(-\infty, a)$
- $\exists\ w_1$ *such that* $l(x) = 0\ \forall x \in (-\infty, w_1)$
- $l(x)$ *is right continuous.*

- $r(x)$ *is a monotonically non-increasing function in* (b, ∞)
- $\exists\ w_2$ *such that* $r(x) = 0\ \forall x \in (w_2, \infty)$
- $r(x)$ *is left continuous.*

Proof: *Try it yourself with hint.*

Hint

1. α-cuts are closed, hence 1_A is closed, added with the fact that, A is normal.
2. Consider the line $\lambda x_1 + (1 - \lambda)x_2$ and observe the membership function.
3. For continuity, assume contrary.
   Construct (a_n) s.t. $x_i < a \quad \forall\ i$ and $lim_{n \to \infty}\ a_n > a$.

Examples

What can be the different types of fuzzy numbers?

- A real number, say *a*

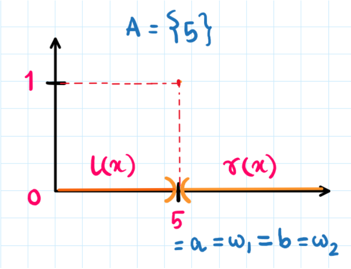

Figure: Real Number as Fuzzy number

- An interval $[a, b]$

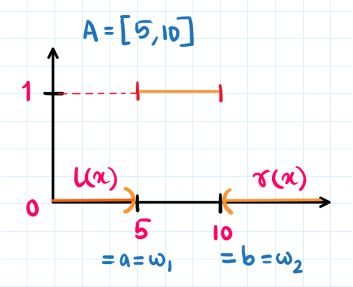

Figure: Interval as fuzzy number

Examples

- Triangular [a b c]

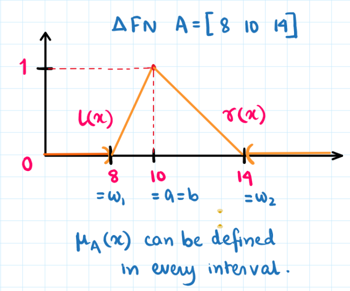

Figure: Triangular fuzzy number

- Trapezoidal [a b c d]

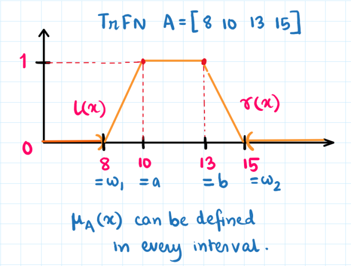

Figure: Trapezoidal fuzzy number

Operations

We will focus on one branch here, which is **Interval based Arithmetic**.

Let $[a, b]$ and $[c, d]$ be two closed intervals. Then:

- $[a, b]^{-1} = \left[\frac{1}{b}, \frac{1}{a}\right]$ (if $0 \notin [a, b]$)
- $[a, b] \pm [c, d] = [a \pm c, b \pm d]$
- $[a, b] \ * \ [c, d] = [\min\{ac, bc, ad, bd\}, \ \max\{ac, bc, ad, bd\}]$
- $[a, b] \ * \ [c, d] = \left[\min\left\{\frac{a}{c}, \frac{b}{c}, \frac{a}{d}, \frac{c}{d}\right\}, \ \max\left\{\frac{a}{c}, \frac{b}{c}, \frac{a}{d}, \frac{c}{d}\right\}\right]$ (if $0 \notin [c, d]$)

Fuzzy Arithmetic works quite similarly, since after all intervals are fuzzy numbers too.

Fuzzy Relations

Relations

A relation represents association or lack of it between element of two sets. Being a generalization of functions, it can be one-to-many (in both ways) but a function can't be so. Mathematically, a relation $\subset$ Cartesian product of sets.

We will be starting with binary relations and move forward till trinary relations in our notation, but firstly we need a mathematical representation of relations.

Even though there are many ways to represent a relation, most common of them as a mapping from domain set elements to codomain set elements (sagittal diagrams), I choose to represent them as a function R defined from the cartesian product of the sets $\otimes S_i$ to $\{0, 1\}$ depending if the element of $\otimes S_i$ belongs to R.

Additionally, for better visuals, I am adding a graph where the axes represents the set elements and we can easily get a 2-d representation of the relation.

Example: Let X represent the set of teachers and Y represents the set of subjects taken up by them. Then R represent the relation between them.

$$X = \{x_1, x_2, x_3\}, \;\; Y = \{y_1, y_2, y_3, y_4\}$$
$$R = \{(x_1, y_1), (x_1, y_2), (x_2, y_3), (x_2, y_4), \ldots\}$$

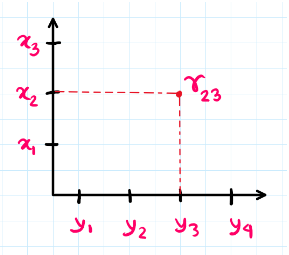

Figure: Binary relation

Define r_{ij} as:

$$r_{ij} = \begin{cases} 1 & \text{if } (x_i, y_j) \in R \\ 0 & \text{if } (x_i, y_j) \notin R \end{cases}$$

Here Domain $\rightarrow X$, and Range $\rightarrow Y$.

Question: How can we generalize this to Fuzzy?

Explanation with an Example

Fuzzy Relation $R : X \rightarrow X$ where $X = \{$Ahmedabad(A), Chennai(C), Durgapur(D), Hyderabad(H), Kolkata(K), Malda(M), Nagpur(N)$\}$ is a crisp set of Indian cities where R describes the concept of distance between them.

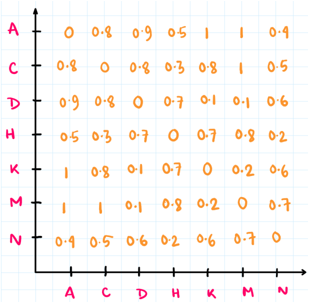

Figure: R representation

$R_{ij} = R_{ji} =$ membership of the cities x_i and x_j in the relation *"distant"*.
Here we use the strength of the relation.

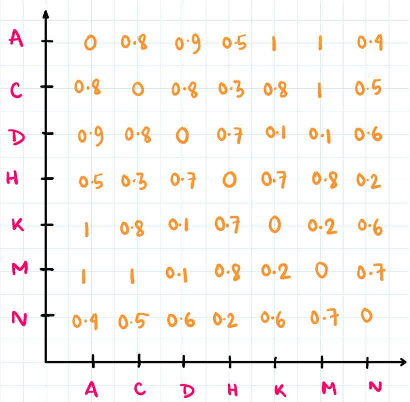

Figure: R representation

Domain and Range can be defined as fuzzy sets on X and Y (Here X) respectively with membership:

$$\mu_{\text{dom}R}(x) = \max_{y \in Y} R(x, y)$$

$$\mu_{\text{ran}R}(x) = \max_{x \in X} R(x, y)$$

$\therefore$ In this case, Range = Domain

$$= \left\{ \frac{A}{1}, \frac{C}{1}, \frac{D}{0.9}, \frac{H}{0.8}, \frac{K}{1}, \frac{M}{1}, \frac{N}{0.7} \right\}$$

Fuzzy Logic

So, assuming we all know *Boolean Logic*, where the truth value of variables are either 0 or 1, *Fuzzy Logic* just generalizes it to have any Real number from $[0, 1]$

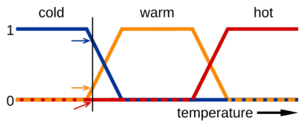

Figure: Fuzzy logic Temperature

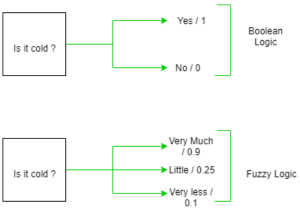

Figure: Fuzzy logic

Fuzzy Graphs

Definition

A fuzzy graph $G = (V, \sigma, \mu)$ is a non-empty set V together with a pair of functions $\sigma : V \rightarrow [0,1]$ and $\mu : V \times V \rightarrow [0,1]$ such that

- $\mu(v_i, v_j) \leq \min\{\sigma(v_i), \sigma(v_j)\} \quad \forall\ v_i, v_j \in V$
- and μ is a symmetric fuzzy relation on σ.

Remember: $V \times V$ represents nothing but the edge set, E.

Example: Let $V = \{a, b, c, d\}$

Defining $\sigma \rightarrow$

$\sigma(a) = 0.5,\ \sigma(b) = 0.9,\ \sigma(c) = 1,\ \sigma(d) = 0.2$

Defining $\mu \rightarrow \mu(a,b) = 0.4,\ \mu(b,c) = 0.5,$

$\mu(c,d) = 0.1,\ \mu(d,a) = 0.1,\ \mu(b,d) = 0.2$

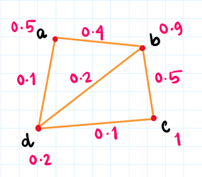

Figure: Fuzzy Graph

Fuzzy Topology

Definition

Let X be a non empty set and $\mathcal{P}(X)$ be the system of all subsets of X. Let $\mathcal{F}$ be the system of all fuzzy sets in X. A pair $\langle X, u \rangle$ is called a *fuzzy topological space* supposing that u is a mapping from the system $\mathcal{P}(X)$ into $\mathcal{F}$ satisfying:

- if $A \subset X$, then $uA(x) = 1$ for all $x \in A$
- if $A \subset X$ contains at most 1 element, then $uA(x) = \psi_A(x)$, where ψ_A is the characteristic function of the set A
- if $A_1 \subset X$, $A_2 \subset X$, then $u(A_1 \cup A_2)(x) = \max\{uA_1(x), uA_2(x)\}$.

The value $uA(x)$ can be interpreted as "the degree of membership of the element x in $\overline{A}$", as $uA(x)$ forms, in fact, a fuzzy set in X.

Fuzzy Clustering

Clustering is an unsupervised machine learning technique that divides the given data into different clusters based on their distances (similarity) from each other.

Definition

Fuzzy Clustering is a type of clustering algorithm in machine learning that allows a data point to belong to more than one cluster with different degrees of membership. Unlike traditional clustering algorithms, such as k-means or hierarchical clustering, which assign each data point to a single cluster, fuzzy clustering assigns a membership degree between 0 and 1 for each data point for each cluster.

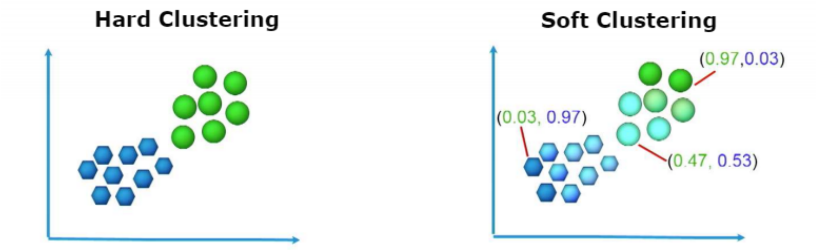

Figure: Fuzzy Clustering

Applications in several fields of Fuzzy clustering : $\rightarrow$

- **Image segmentation:** Groups pixels with similar properties like color or texture.
- **Marketing:** Segments customers based on preferences and behavior for targeted campaigns.
- **Environmental monitoring:** Identifies areas of concern by clustering similar pollution levels.

Open Research Directions

Research directions in fuzzy theory: $\rightarrow$

- Quantum fuzzy logic.
- Integration with deep learning, functions like IoT and cybersecurity.
- Hierarchical Fuzzy Granulation
- Hybrid Models with Probability and Statistics

References

[1] Prof. Niladri Chatterjee. *NPTEL :: Mathematics - NOC:Introduction to Fuzzy Set Theory, Arithmetic and Logic*. URL: https://archive.nptel.ac.in/courses/111/102/111102130/.

[2] Malik Matthew Mordeson. "Fuzzy Graph Theory". In: Springer, 2018. Chap. 2.2.

[3] L.A. Zadeh. "Fuzzy sets". In: *Information and Control* 8.3 (1965), pp. 338–353. ISSN: 0019-9958. DOI: https://doi.org/10.1016/S0019-9958(65)90241-X. URL: https://www.sciencedirect.com/science/article/pii/S001999586590241X.

The End!

Thank You!